The Felix Klein *Protocols*

Eugene Chislenko and Yuri Tschinkel

All photographs: Mathematisches Institut Universität Göttingen.

Felix Klein

The Göttingen Archive

Two plain shelves in Göttingen, in the entrance room of the *Mathematisches Institut* library, hold an astounding collection of mathematical manuscripts and rare books. In this locked *Giftschrank*, or "poison cabinet", stand several hundred volumes, largely handwritten and mostly unique, that form an extensive archive documenting the development of one of the world's most important mathematical centers—the home of Gauss, Riemann, Dirichlet, Klein, Hilbert, Minkowski, Courant, Weyl, and other leading mathematicians and physicists of the 19th and early 20th centuries. A recent *Report on the Göttingen Mathematical Institute Archive* [2] cites "a range of material unrivalled in quantity and quality": "No single archive is even remotely comparable", not only because Göttingen was "the leading place for mathematics in the world", but also because "no other community has left such a detailed record of its activity …usually we are lucky to have lecture lists, with no indication of the contents."

The collection runs from early handwritten lectures by Dirichlet, Riemann, and Clebsch, through almost 100 volumes by Hilbert, to volumes of Minkowski, Hasse, and Siegel on number theory, Noether on algebra, and Max Born on quantum mechanics. But the largest and most impressive of its centerpieces is the *Seminar-Protokolle* of Felix Klein: a detailed handwritten registry, spanning over 8,000 pages in 29 volumes, of forty years of seminar lectures by him, his colleagues and students, and distinguished visitors. These *Protocols*, previously unpublished, are now available digitally, as part of a project sponsored by the Clay Mathematics Institute. They constitute one of the richest records of mathematical activity in modern times.

Felix Klein

Felix Klein was one of the central figures in 19th and early 20th century mathematics. Born in Düsseldorf, Germany, in 1849, he studied in Bonn under Plücker, and then worked briefly in Göttingen under Alfred Clebsch and with the young Sophus Lie in Berlin and Paris. Plücker's sudden death and Clebsch's encouragement left him with unfinished projects in geometry, where he made his earliest and most lasting creative achievements. His first major result was the construction of the Klein model of non-Euclidean geometry, establishing that the consistency of non-Euclidean geometry is equivalent to the consistency of Euclidean geometry. This put an end to the long controversy concerning the legitimacy of non-Euclidean geometry. After a brief military service in 1870 and his *Habilitation* in Göttingen in 1871, he "started lecturing on 'Optics' and 'Interactions of Natural Forces', without having studied much physics" [5], p. 1. In 1872, the twenty-three-year-old Klein began

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Figure 1. The *Protocols*.

his first professorship at the University of Erlangen. The main themes of his inauguration speech were: the role of mathematics in the system of the sciences and in society, practical applications, and above all, "the general purpose of mathematics education, and especially the form we aspire to give to it at our universities" [4], p. 4. While at Erlangen he developed his revolutionary *Erlangen Program*, unifying the various geometries of the time by classifying them according to their corresponding groups of transformations. Over the next decade, he continued to do groundbreaking work in group theory, function theory, and related areas. But he still intended "to return one day to physics, and …to science in general", and even worked in a zoology lab for one semester [5], p. 2.

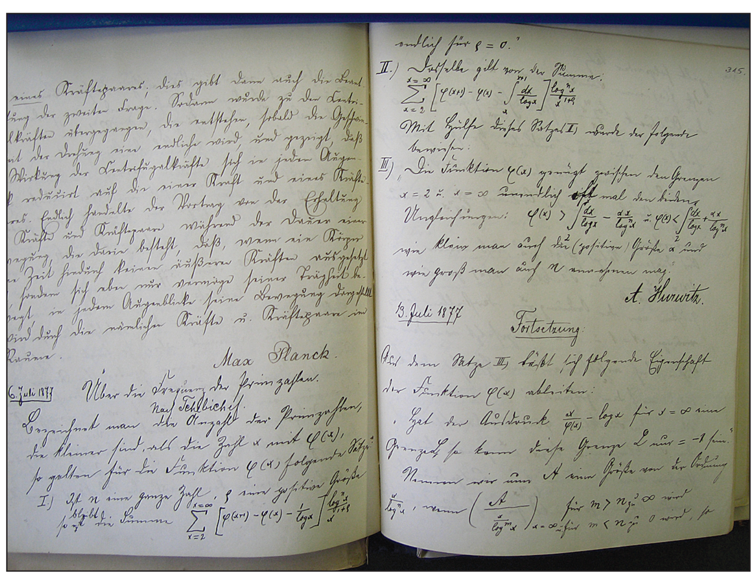
In 1875 Klein married Anne Hegel, granddaughter of the philosopher G. W. F. Hegel. (His notes record the "beginning of an ordered existence.") They moved to Munich, where Klein's duties included the teaching of engineers, which, "since I grew up in an industrial area, called up the most cherished memories of my boyhood" [5], p. 2. In addition he organized geometry classes for future teachers, and "tended a small flock of gifted mathematics students" including Hurwitz, Planck, and Ricci (Figure 2 shows a *Protocol* page with summaries, in each student's own handwriting, of presentations by Planck and by Hurwitz on the distribution of prime numbers). He moved again in 1880, this time to Leipzig in Saxony, where his most creative period would come to an end. Overstrained by an excessive workload and intense competition with Henri Poincaré over automorphic functions, he collapsed in 1882, unable to keep up his series of groundbreaking discoveries:

> Decisive illness: Overexertion as underlying cause. Total breakdown of serious productivity. Impossibility of carrying out academic and organisational work alongside general teaching activity with equal energy… Image of the coat that is too wide for me [6], p. 2.

He would never quite regain the brilliant mathematical creativity of his early years, and in his notes from 1883 he wrote: "My great productivity is entirely over" [8].

Soon, however, Klein entered a second period of great productivity, this time longer and of a different kind. In the fall of 1885 he received a "call" from the University of Göttingen and accepted immediately. In his private notes, he summarized the advantages: "house with garden, lighter duties, Prussia…. Seeking: concentrated scientific existence on the basis of a sensible family life" [8], pp. 4–5.

The Prussian education ministry's attitude at the time was the following ([10], pp. 23–24):


Max Planck

6. Juli 1877

A. Hurwitz

Fortsetzung

Figure 2. Hurwitz: On the distribution of primes.

What was previously often impossible, is in Prussia always feasible, with insight, gentleness and love for the education of German youth and love for the nurture of our hallowed science and spiritual labor. … [We have] to confront the caste-egoism of the cliquishly bonded faculty with an iron fist at the demand that their teaching be left to their own discretion without regard for the curricular requirements of their listeners, and at the same time deny the request that all disciplines be represented at all universities for the sake of student attendance. On the contrary—certain subjects should only be covered at one or two German universities, and there exceptionally well. It is and remains nonsense to lecture on Romance philology twenty and more times in front of two or three listeners…. It is also no less inconsiderate of the fame-seeking faculty towards the students to call every small new nuance a new science and even to endow new chairs for it, let alone at several universities.

Prussian education minister Friedrich Althoff, Klein's comrade from their army days in 1870, supported Klein's vision of reviving the great mathematical tradition developed in Göttingen by Gauss, Riemann, and Dirichlet earlier in the century. It was thus decided to concentrate German mathematics and physics in Göttingen. Klein recognized the extraordinary talent of the young David Hilbert and, with Althoff's support, succeeded in hiring him in 1895. Then, in 1902, the two arranged for the establishment of a third chair in pure mathematics, unprecedented in German universities, for Hermann Minkowski. Klein, Hilbert, and Minkowski lived in Göttingen for the rest of their lives, leading a mathematical community that for many was the foremost in the world.

During his tenure in Göttingen, Klein founded the Göttingen Mathematics Association, built up the *Mathematische Annalen* (a journal started by Clebsch) into the leading mathematics journal in the country, edited the monumental *Encyclopedia of the Mathematical Sciences and their Applications*, and played a leading role in Germany's first national association of mathematicians. He cultivated contacts with many leading mathematicians abroad, inviting them to visit as well as traveling himself. He made repeated trips to the United States, and his Evanston Colloquium lectures at the World's Fair in Chicago became a great inspiration to the still nascent American mathematical community. He organized funding for a growing stream of talented American mathematics students to study with him in Göttingen, among them six future presidents of the American Mathematical Society. Klein himself was left deeply impressed by his experience in the United States ([5], p. 4):

The World's Fair in Chicago in 1893 (where I was sent as commissar of the Ministry of Education) gave me powerful new impulses…. I allowed the conditions confronting me, especially the peculiar American system of education, to make an impression on me. I returned with the vivid conviction that it is our most urgent task to establish direct connections between our university operations and the controlling powers of practical life, first and foremost technology, but also the pressing questions of the general system of education…. I therefore mainly abandoned my own academic work from then on, and directed all of my energy toward establishing a cooperative interaction with others.

Klein began to take an increasing interest in the applications of mathematics and in education reform. By 1898 he had formed the Göttingen Association for the Advancement of Applied Physics and Mathematics, an association that succeeded in doubling the number of professors in mathematics and physics and raised enough funding to establish the Institutes for Applied Electricity, Applied Mathematics and Mechanics, and Geophysics. Among the new faculty were Carl Runge, Ludwig Prandtl, and Emil Wiechert. At the same time, Klein campaigned vigorously for a program of education reforms that became known as the Klein reforms. These included the introduction of the basic concepts of calculus in secondary schools, a lasting change in many countries. In 1908, the International Congress of Mathematicians, in which Klein had played a prominent part for many years, elected him as the first president of the newly founded International Commission on Mathematical Instruction, an organization still active today. He served in that capacity for several years and oversaw several series of publications associated with the commission.

Throughout his career, Klein was a legendary teacher. Harry Walter Tyler, an American who studied with Klein soon after he left Leipzig, wrote back enthusiastically to William Osgood:

I know of no one who can approach him as a lecturer…. He's certainly acute, fertile in resource, not only understands other people, but makes them understand him, and seems to have a very broad firm grasp of the philosophical relations and bearings of different subjects, as well as great versatility and acquaintance with literature.

This opinion was shared by many of his students, and many of them went on to become prominent in their own right. Among his more than 50 doctoral students are Cole, Fine, Fricke, Furtwängler, Harnack, Hurwitz, Ostrowski, and van Vleck.

Klein retired in 1913, his career spanning almost the entire period of the German Empire, and died in Göttingen in 1925.

The Mathematical *Protocols*

Klein conducted his seminars in both semesters of every year, from the summer semester of 1872 until his retirement in 1913. Presentations made in the seminars were painstakingly recorded in the *Protocol* books, usually by the speaker, just as Göttingen mathematics lecture courses were recorded in other notebooks and placed in the library for students' reference. Klein would later describe the development of his ambitious teaching and research program in his Evanston lectures:

> As regards my own higher lectures, I have pursued a certain plan in selecting the subjects for different years, my general aim being *to gain, in the course of time, a complete view of the whole field of modern mathematics, with particular regard to the intuitional or* (in the highest sense of the term) *geometrical standpoint* [7], p. 96.

Klein's drive toward completeness made him, along with Hilbert and Poincaré, the last of the mathematicians who could claim to have a grasp of the entire field. But his description is of his lecture courses, not of his seminars, for whose breadth and ambition this would be a clear understatement. His forty years of seminars not only covered the major branches of the field, but expanded into mechanics, astronomy, geodesy, hydrodynamics, electricity, elasticity theory, and in the last years before his retirement, the psychology and teaching of mathematics. No wonder the presentations filled over 8,000 pages! David Rowe wrote about the *Protocols*: "Although it would appear that few have perused them since they were first placed there, they are undoubtedly the best single source documenting the rich panoply of ideas that characterized Klein's teaching activity" [13], p. 34 n. 5.

The seminars from Erlangen and Munich show little unity of subject matter. Most of the 1870s seminars are catalogued as "Seminar on Various Topics" or "Seminar on Various Topics in Geometry and Algebra", and the entire record of that decade is contained in the first *Protocol* volume, most of whose entries are summaries a page or two in length. The constant change of topics from week to week can be seen in the presentation list in Klein's first seminar, taught jointly with Clebsch in Göttingen just before Klein's move to his first professorship. The presentations are on:

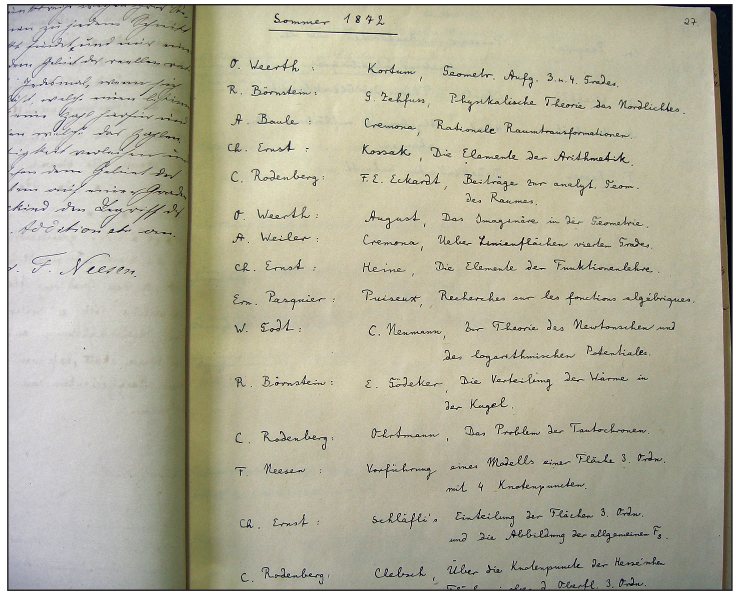


Figure 3. First seminar: Table of contents.

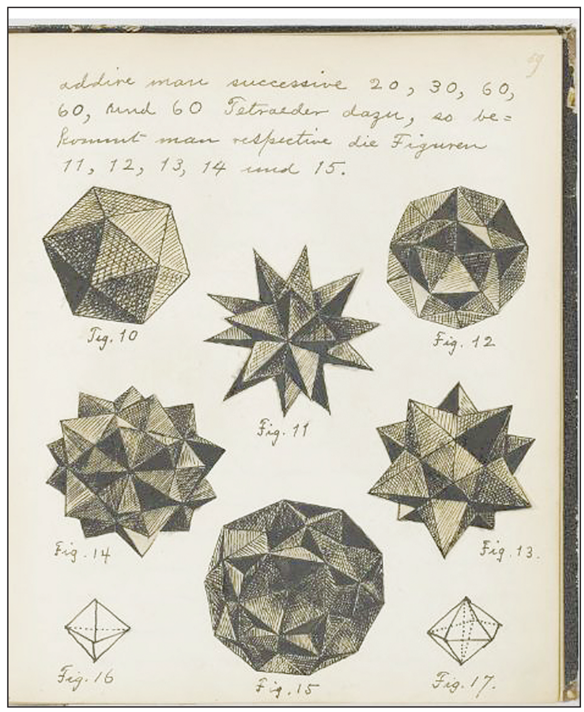
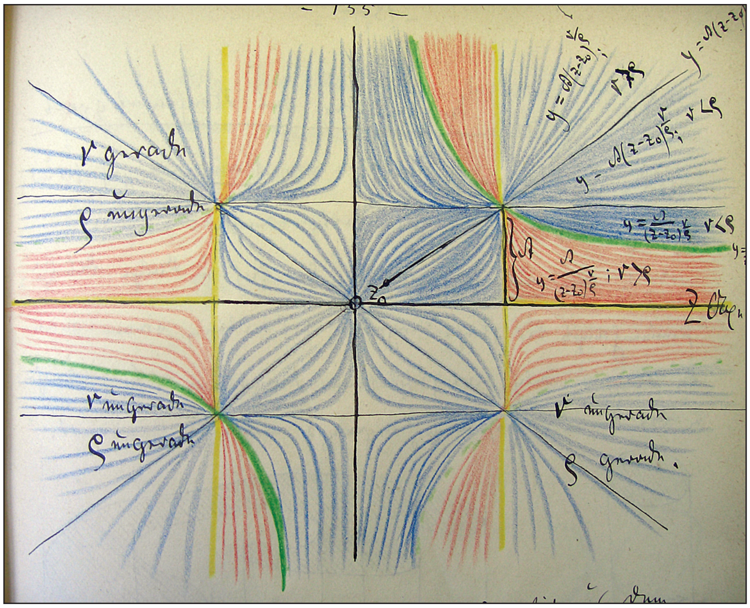


Figure 4. Report on regular solids in 4-dimensional space, November 29, 1880.

- Geometric Problems of the 3rd and 4th Degree,
- The Physical Theory of the Northern Lights,
- Rational Transformations,
- The Elements of Arithmetic,
- Contributions to the Analytic Geometry of Space,
- The Imaginary in Geometry,
- On Scrolls of Degree 4,
- The Elements of Function Theory,
- Investigations on Algebraic Functions,



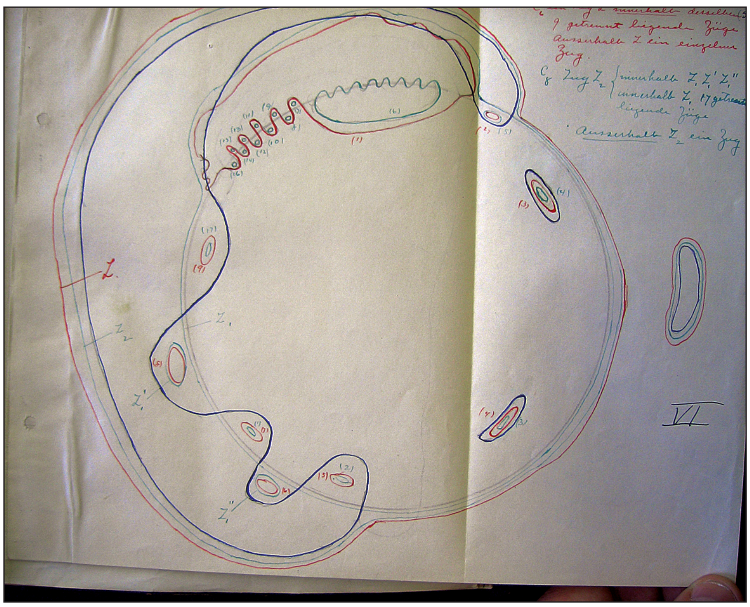


Figure 5. Curves.

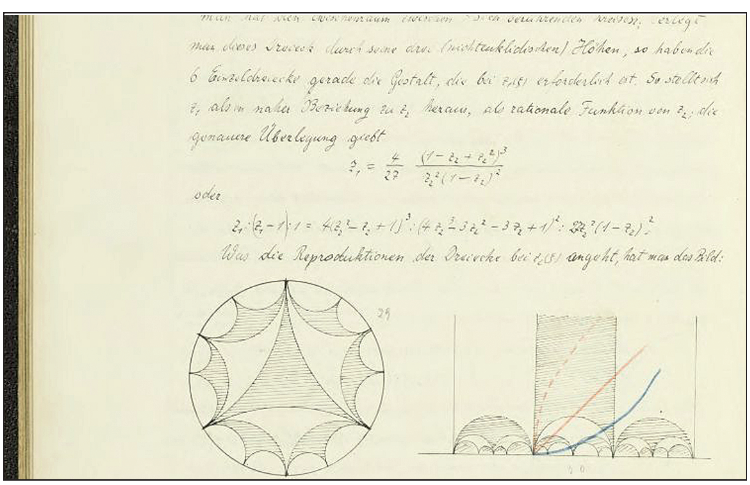


Figure 6. Klein, Hilbert, Minkowski, winter 1905–06.

- On the Theory of Newtonian and Logarithmic Potential,
- The Distribution of Heat in a Sphere,
- The Tautochrone Problem,

and so forth (see Figure 3).

In May 1875 Klein reports on solutions of polynomial equations of degree 5 via elliptic functions; the seminar of 1876 has talks on magnetic curves, reflected light, and the effects of an electric point on an isolated metallic sphere. The seminars of 1877 contain notes on elastic strings, the distribution of heat in solids, Ampère's and Ohm's laws, branching of electric current, and polarized light. The seminars of 1879 have a lecture on the 27 lines of cubic surfaces, reports on Fresnel's wave surface, methods of enumerative geometry, and modular curves.

Soon the seminars become more focused. The seminars of the 1880s are almost exclusively on Klein's own research topics in function theory and group theory. The winter 1882–83 seminar, for example, deals with "Hyperelliptic, abelian and theta functions" and the winter 1885–86 seminar covers "Hyperelliptic functions and the Kummer surface".

The early Göttingen years were a period of transition ([5] p. 3):

> In the presence of Schwarz, there was—I might say: fortunately—at first no way to make a wider impact on the multitude of students in Göttingen. But I used the first years in Göttingen, of course partly to continue my previously started work, but then especially to fill in the gaps in my mathematical-physical training, of which I was vividly aware, and which I had not been able to correct in my previous years of overwork.

Klein's colleague Schwarz had not yet made way for Hilbert, and Klein did not yet feel himself to have enough freedom or experience to rebuild his department as he saw fit. For the next ten years, from 1886 to 1896, he continued to conduct his seminars alone, and though he gave several lecture courses on mechanics, the seminars remained mainly in pure mathematics. In the summer of 1892 the seminar was devoted to number theory: distribution of primes, Diophantus and his works, quadratic and biquadratic reciprocity, reduction theory of quadratic forms, class numbers, representation of integers by quadratic forms (Furtwängler), and composition of quadratic forms and complex multiplication (Epstein). Perhaps the last several of these seminars, as Klein insisted, "should be seen only as offshoots of my activity before 1892" [5], the year Klein intensified his interest in other fields. But "the change", as Klein describes it, "did not arrive in one fell swoop" [5].

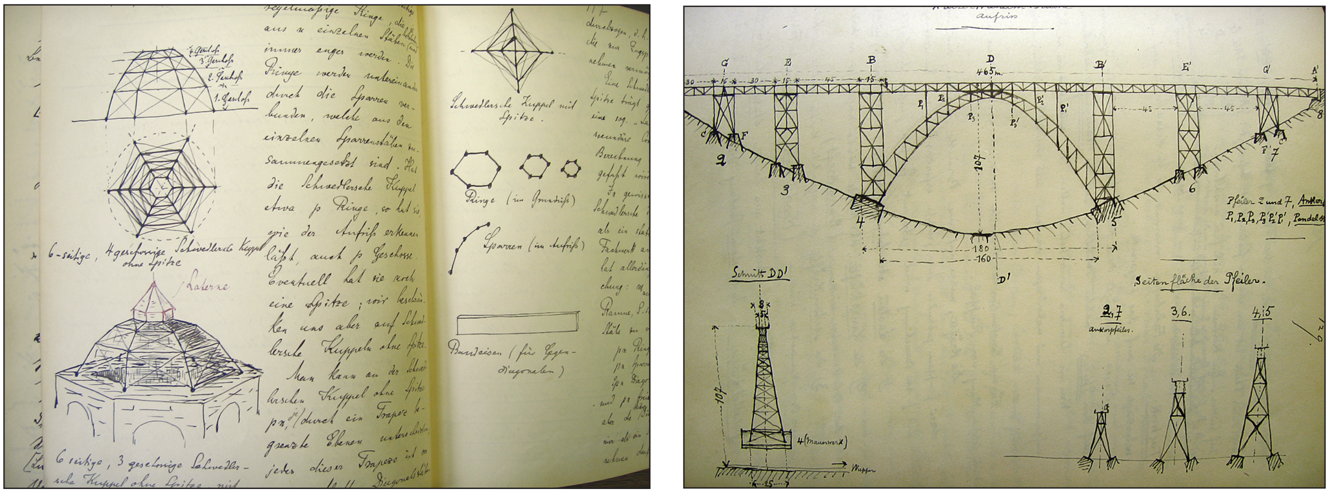


Figure 7. Domes and bridges.

Among the mathematical *Protocols*, an especially interesting seminar is the one conducted by Klein, Hilbert, and Minkowski in the winter 1905–06: Klein lectured throughout, with Töplitz taking notes (see Figure 6). The topic is automorphic functions, with the aim of reporting "coherently" on "my [Klein's] early efforts in this direction and some of the still unsettled questions, as well as on …the progress achieved by Poincaré and its relation to my own ansatz".

The Later *Protocols*: Application and Education

The last fifteen years of *Protocols*, from 1897 to 1912, show expansion in every sense. They fill over half of the *Protocol* volumes, go far beyond the range of subjects in the earlier seminars, and include many collaborations, at a time when Klein was at the peak of his powers as a leader of the international mathematics community. After teaching alone from 1872 to 1896, Klein taught four seminars in a row with Hilbert, and by 1909 he would co-teach five seminars with the physicist Karl Schwarzschild, six together with Ludwig Prandtl and Carl Runge, and, in 1905–7, a series of four seminars with both Hilbert and Minkowski on differential equations and automorphic functions. Meanwhile Klein, having warned against "the danger of a separation between abstract mathematical science and its scientific and technical applications" ([7], p. 50), had also begun to place more and more emphasis on bridging mathematics and the other disciplines. Earlier seminars had already included presentation topics such as "Vibrations of a Violin String", in the winter semester 1877–78, "The Theory of Billiards" in the summer 1887 seminar on the theory of tops, and "The Calculation of Death Charts" in the summer 1893 seminar on probability theory. But these isolated presentations were still the exception. In 1898, after several years of seminars on pure mathematics, Klein and Hilbert jointly taught two seminars on mechanics, with presentations ranging from the more standard theoretical topics to "On the Bicycle" and "On the Theory of Billiards". The summer 1900 seminar "Technical Applications of Elasticity Theory" contains some of the *Protocols*' most meticulously illustrated entries, including presentations on cupolas and on bridges (see Figure 7). A subsequent mechanics seminar in the winter of 1901–02 includes a presentation "On Seismographs", and the winter 1900–01 seminar "Applications of Projective Geometry" includes reports on "Hermann Ritter's Perspectograph", "Hauck-Mauer's Perspective-Drawing Apparatus", "On Painter's Perspective", and "Stereoscopic Vision".

Looking through the volumes from the turn of the century, one finds a recurring interest in ships. The winter semester 1899–1900 seminar, devoted to "The Theory of Ships", includes presentations ranging from fairly abstract to surprisingly specific: "On the Stable Balance of Swimming Bodies", "On the Stability of Ships", "Spatial Contents and Sinking of a Ship", "Sails and Rudders", "Ship Waves in a Canal", and even "On the 'Seiches'", a

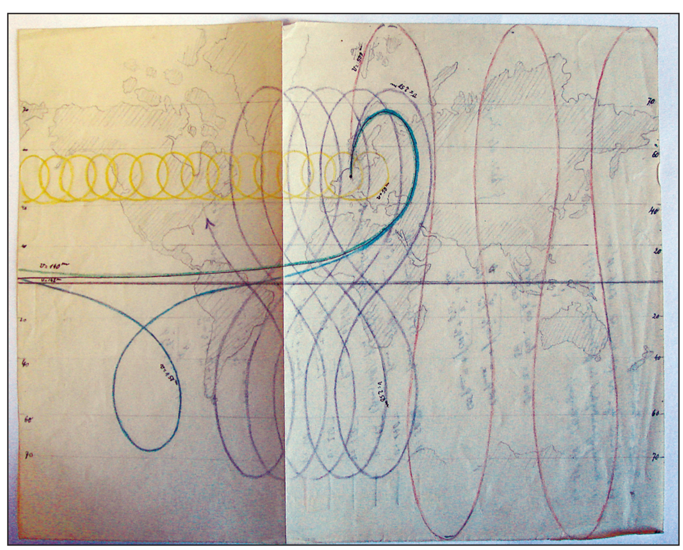


Figure 8. Jetstreams.

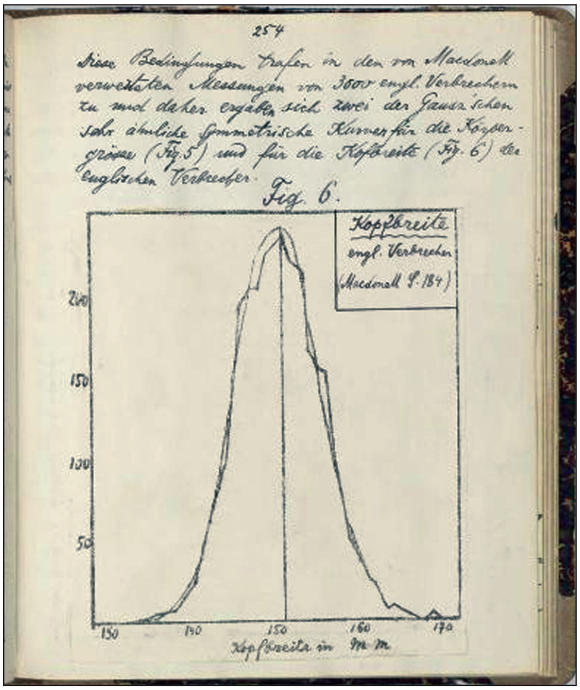


Figure 9. [3], v. 14 p. 254: Head width of English criminals. The discussion starts (on an earlier page) with: The speaker wishes to emphasize that it is quite important to let the facts speak for themselves, before approaching them with any particular interpretation. Since several talks of theoretical content now follow, the speaker believes himself to carry out a rewarding task by reporting straightforwardly on several statistical and anthropological observations from the geometric point of view. The observations concerning humans are twofold: on the one hand as beings endowed with a differentiated psychical capacity, and on the other hand as objects of zoological research in the broadest sense of the word.

discussion of the occasional sudden changes in the water level on Lake Geneva, developing formulas with constant frequency and decreasing amplitude and showing their accord with empirical observations on the lake's shore. The winter 1901–02 seminar on mechanics includes a presentation on "Ship Resistance in a Calm Sea", and the winter 1903–04 seminar on hydrodynamics has one "On Turbines". The entire winter 1907–08 seminar, co-taught with Prandtl, Runge, and Wiechert, is a "Hydrodynamics seminar with special attention to the hydrodynamics of ships". Its presentations on more general topics in hydrodynamics give way to "Ship Waves", "The Theory of Ship Propellers", and "On Ship Resistance in Unbounded Water", continuing through the first half of the next semester with "Ship Oscillation", "Continuation of the Presentation on Ship Resistance", "Lounz's Theory of Turbines", and "Ship Resistance in Canals". One wonders how much the introduction of such topics into the nation's leading mathematics department had to do with the rapid buildup of the German navy in the decade before World War I.[1]

Along with ship theory, the turn-of-the-century seminars show an increased emphasis on air and space. These topics can be found occasionally in the earlier volumes, starting in the very first pages of the *Protocols* with an 1872 presentation on the "Physical Theory of the Northern Lights". But no significant part of any seminar before 1900 is devoted to them. The summer 1902 seminar, by contrast, has astronomy as its general theme, and includes many presentations treating the moon and orbits. The summer 1908 seminar "Dynamic Meteorology", whose first half is the continuation of the winter seminar on ships, includes presentations on the "Thermodynamics of the Atmosphere", treating topics such as humidity, the mixing of air masses, and cloud formation; "The General Circulation of the Atmosphere"; "Cyclones"; and "Helmholtz's Treatment of Atmospheric Movement".

The summer seminar 1911 (joint with Bernstein) was devoted to insurance mathematics, i.e., "death charts" and biometrics, a reccurring theme in the *Protocols*.

The final *Protocol* volume contains four seminars, three of them devoted to the psychology and education of mathematics. By this time Klein was running the International Commission on Mathematics Instruction, and these seminars ran partly in parallel with his efforts there. The winter semester 1909–1910 seminar is titled "Mathematics and Psychology", and listed on the volume's cover as "Psychological Foundations of Mathematics". Klein states in his opening speech to the seminar:

> The general topic is the intersection points of *mathematics* and *philosophy*. The more strictly *logical* questions will be treated in the parallel lecture course by Zermelo; here we shall discuss all of the other mental processes which accompany the logical processes and in part precede them, and which will here be called simply *psychological*.

Klein spends most of his opening lecture laying out a set of suggested presentation topics, which give an indication of his nontechnical interests at the time. He puts forward, with descriptions, six central themes:

1. On the working methods of productive mathematicians

[1] *Berlin's new navy laws, passed in 1898 and 1900, envisioned Germany as a naval superpower equal to Britain, and prefigured an enormous growth in ship production through 1914. See, for example, John Maurer's article on the Anglo-German naval race in the bibliography below, with a discussion of the "dramatic growth of German naval power between 1906 and 1914" ([9, p. 287]).*

2. On the development of basic mathematical intuition in the growing individual
3. The formation and epistemological importance of mathematical axioms
4. On the errors of mathematicians
5. Implications for mathematical instruction
6. On the position of mathematics in the sciences.

The seminar includes many reports and informal discussions, recorded in summary form by Klein himself, on these and other topics, including Klein's recounting of his own mathematical development as well as that of Gauss, Lie, and other mathematicians. Here is one episode from Klein's early years (written down by Klein himself in the third person) [3], v. 29, pp. 7–9:

> Klein had learned the projective way of thinking from Plücker and Clebsch, and had then read Cayley's paper with great enthusiasm in the autumn of 1869. Then in the winter of 1869–70 (in Berlin) he heard from Stolz, who was studying with him there, about the existence of non-Euclidean geometry. It was immediately *self-evident* to him that the two would have to be in correspondence with each other. He presented this view in February 1870 in Weierstrass's mathematics seminar, at the end of a lecture on Cayleyan measurement, in the form of a question. But Weierstrass retorted that these were completely separate areas of the science. After that Klein abandoned the idea for the time being.
>
> It reemerged for him as he was once again with Stolz in the summer of 1871 (this time in Göttingen). Stolz gave him details from Lobachevsky, von Staudt, Beltrami (whom Klein had not read at all at that point; even today he knows them very inadequately). There was everywhere a correspondence with the correctly understood Cayleyan doctrine. On the other hand strong suppression by the view, coming especially from Lotze, that the entire non-Euclidean speculations were nonsense. Out of this back and forth there grew the first publication, which appeared in short form in the *Göttinger Nachrichten* of August 1871 and in full soon thereafter in *Mathematische Annalen* 4.
>
> The paper in *Annalen* VI (1872) shows the great resistance that the line of thought encountered in mathematical circles. Even Cayley has never been able to bring himself around to full agreement. He said at the 1873 meeting of the British Association in Bradford that he views the parallel axiom as "strictly axiomatic", and in Vol. II, p. 605 of his collected works he again remarks that a grounding of the concept of distance in von Staudt's projective coordinate system gives rise to at least the appearance of circular reasoning.
>
> Here, then, is an example in which a mathematical insight is first so to speak pre-formed in an individual, and then, as a result of the resistance it encounters, is felt by the individual to be an advance and is worked out clearly from all sides in a fight against all kinds of objections.
>
> What happens next is that a new generation adopts the result from the start as axiomatic, no longer understands the earlier differences of opinion, and more or less goes back to normal about the whole thing.

Klein, who had much more to say on these topics, made ten of the seminar's presentations himself, and often commented on other presentations and on his own creative process. He summarized his intuitive approach and emphasis on breadth with an example from the theory of functions:

> As for my own work, I have often proceeded in such a way that I viewed the results of two subareas as given and asked what the one means for the other. Compare as typical the use of algebraic invariant theory in my introduction of hyperelliptic and abelian functions …In stating the corresponding theorems, I have let myself be guided in many cases by an indeterminate but, with hindsight, accurate feeling of analogy. I took a special pleasure in this: I did not quite know which invariant of a binary form Sylvester had called a catalecticant, but I reckoned that the first term in the series expansion of certain hyperelliptic Sigmas must be exactly this catalecticant. It was Hilbert who helped me to put things straight, but the theorem, as I had suspected, really was correct.

This intuitive method of analogy did not meet with universal approval. At the 1900 International Congress of Mathematicians in Paris, Poincaré commented on Klein's approach as follows ([11], p. 116):

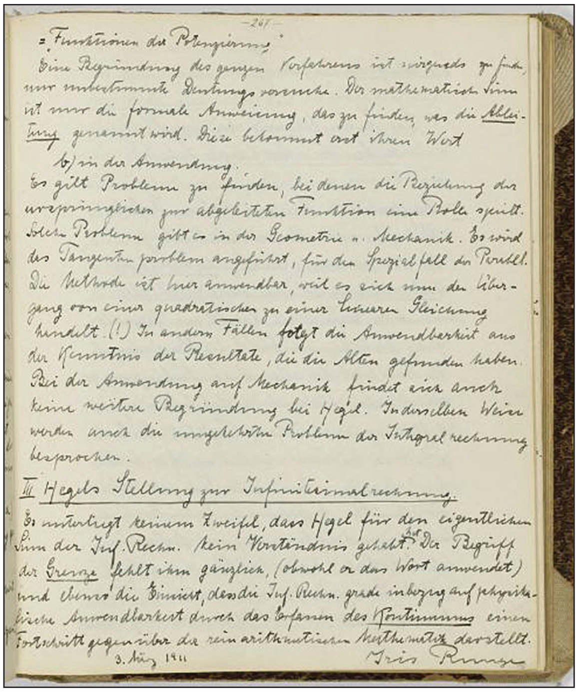


Figure 10. From a sharply critical presentation by Iris Runge, daughter of Carl Runge, on "Infinitesimal Calculus in Hegel": It is beyond any doubt that Hegel had no understanding of the actual meaning of the infinitesimal calculus. He entirely lacked the concept of limit (although he uses the word) and likewise the insight that the infinitesimal calculus represents an advance over purely arithmetic mathematics, especially with respect to physical applications, through its grasp of the continuum.

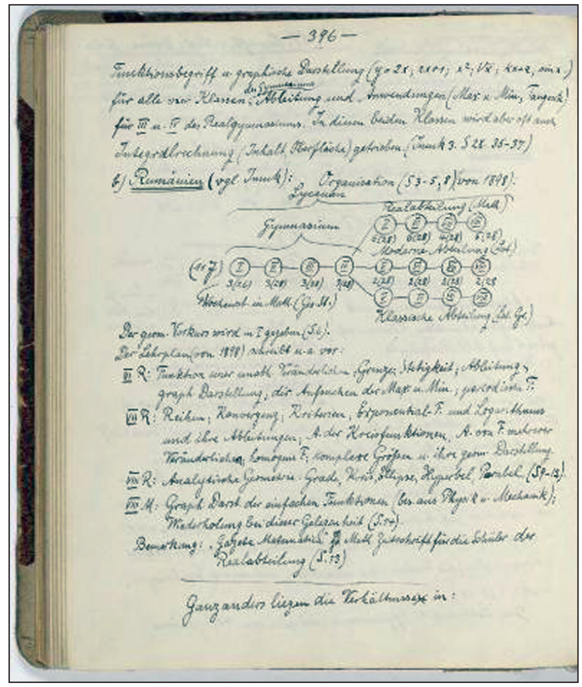


Figure 11. The high school mathematics curriculum in Romania, 1898: VII (age 17), 4 (out of 28) hours of mathematics per week: Series, convergence criteria, the exponential function and logarithm and their derivatives, derivatives of functions of several variables, complex numbers and their geometric representation.

> Look on the other hand at Mr. Klein: he studies one of the most abstract questions in the theory of functions, namely, whether, given a Riemann surface, there is always a function that admits some prescribed singularities: for example two singular logarithmic points with equal residues and of opposite sign. What does the renowned German geometer do? He replaces the Riemann surface by a metallic surface whose electric conductivity varies according to certain laws. He puts the two logarithmic points in contact with the two poles of an electric source. The current will have to pass through, and the manner in which the current is distributed over the surface will define a function whose singularities will be precisely those predicted by the claim.
>
> Without a doubt, Mr. Klein knows full well that he has only given a sketch: nevertheless he has not hesitated to publish it; and he probably expected to find in it if not a rigorous demonstration, then at least some sort of moral certainty. A logician would have rejected such an idea with horror, or rather he would not have had to reject it, since in his mind it could never have been born.

The tension left over from the competition over automorphic functions in the 1880s had not dissipated, and Poincaré's name is conspicuously absent in the psychology and philosophy seminar, despite his series of books on its topics published in the preceding decade. But many other thinkers do make an appearance, among them Aristotle, Kant, Goethe, Schiller, Hegel, Comte, and Spencer.

Klein's students and assistants made presentations including Bernstein on Cantor, Weyl on the role of mathematics in the system of the sciences, Errera on the inner ear and spatial perception, and the occasional pearl of anthropological wisdom:

> Steckel relates some of the observations he believes himself to have made in the East concerning the conduct of members of different races (Germans, Jews, Poles) with regard to mathematical subject matter: Germans calculate $7\frac{1}{4} - \frac{3}{4}$ in the form $= 7 - \frac{1}{2} = 6\frac{1}{2}$, thus grasping the task intuitively; Jews calculate $7\frac{1}{4} = \frac{29}{4}$, therefore $7\frac{1}{4} - \frac{3}{4} = 26/4 = 6\frac{1}{2}$, thus applying general logical rules. Poles

> tend to grasp only the words of the mathematical rules, which is how they then excel in language instruction as well. [3], v. 29, pp. 19–20.

The winter 1910–1911 seminar covers mathematics education, running parallel to Klein's lecture course on the modern development of mathematics education and focusing especially on elementary schools. This seminar is one of the signs of his interest not only in universities, but in education at all levels. Several short presentations, mostly on various aspects of mathematics education in elementary schools (29:76–96), are followed by a unified series of lectures on "Teacher Education", providing a tightly structured overview of the current structure and problems of training mathematics teachers for elementary schools. Rounding out the seminar are similar overviews of conditions in vocational schools, in girls' schools, and in Austria.

The last seminar in the *Protocols*, organized by Klein but led during his illness by his former student Rudolf Schimmack, is an ambitious survey of the state of mathematics education across Europe (summer 1912). The presentations on Germany tend to compare the current system to the past and to other countries. "To What Extent is Euclid's Teaching Continued in Today's German Schoolbooks?" compares Euclid's presentation of his material with contemporary textbooks, particularly the influential ones by Kambly and Berendsen-Gölting, describing a fading but still very noticeable Euclidean influence. "On the Reform Movement in Germany" provides a historical overview of the mathematics education reform movement and an alternative report on Berendsen-Gölting's textbook. There is also a "Comparison of the Organization of Higher Learning in Germany and France", and many presentations on individual foreign countries. England and France receive special attention. "The High School System and Traditional Euclid Lessons in England" describes the structure, history, and recent reforms of English schools and of methods of teaching Euclid, including a discussion of textbooks and examination questions, and is followed by a second report on "Recent Reforms of Geometry Education in England". "The Reform Movement in France" describes a movement away from abstraction and toward concrete examples and applications, and an emphasis on the concept of function. "The New Form of Geometry Education in France" treats in more detail Méray's textbook *Nouveaux éléments de géométrie*, his innovations in treating displacement, translation, rotation, and other basic notions, and his influence, and criticizes him for what it describes as a counterintuitive approach and a lack of economy in treating axioms. Other seminar participants cover "The Question of Geometry Education in Italy", "Reform Efforts in Mathematics Education in Hungarian Middle Schools", and "The Reform Movement in Arithmetic and Algebra Education in the United States and England", and there are also "A Sketch of the School System in Switzerland", a bleak report "On the Organization of Schools in Russia", and a more hopeful report on Finland. "The State of Reform Efforts in Mathematics Education in Some Other Nations" summarizes some advances in Sweden and Romania, but laments that Belgium remains backward in many respects, and Holland "hardly better". Germany receives its share of comparative criticism as well; "The Intuitive Design of Basic Geometry Education", for example, advocates a reorganization of German geometry education into a two-tier approach following that of Austria, so that students who do not reach the higher levels of education still have a thorough and intuitively comprehensible overview of geometry.

Klein's central role in the international reform of mathematics education lends an added importance to the records of his own discussions of education with his students and associates. But it also lends an added importance to the entire set of *Protocols*. While Klein spoke of attaining a view of the whole of mathematics, by the end of his career he had a nearly complete view of mathematics education as well, having toured the school and university systems of many countries and spoken with the leading educators of his time. The gradual rethinking and development of his seminars' structure and scope, of how he ran the seminars, how he assigned topics, what kinds of participation he encouraged, were the result of decades of the most serious and influential thinking about teaching itself. The *Protocol* volumes are, among other things, a career-long, step-by-step record of the development of one of the great modern educators. Even the mere existence of these 29 volumes is a monument to taking teaching seriously, and to believing in the importance of what one's students say.

The *Protocols* themselves have never been published or extensively studied. A recent initiative, supported by the Clay Mathematics Institute, has used the latest in scanning technology to digitize the complete *Protocols* in November of 2006, and to make them available on the internet. Jim Carlson reports on the project in the accompanying sidebar.

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Digitizing the Classics

One thing that modern technology makes possible is public access to the classics of mathematics. Archives such as JSTOR have posted mathematics journals dating back to the 18th century, but they have left largely untouched other important documents—lecture notes, manuscripts, and work books. The Clay Mathematics Institute (CMI) has recently undertaken several initiatives in cooperation with other institutions to digitize and disseminate some of the most important of these. In all cases, CMI has provided some or all of the funding, and in some cases it has helped to organize the work. The first of these projects was the digitization of the oldest extant complete copy of Euclid's *Elements*. This is the d'Orville manuscript, dated to 888 A. D., which was acquired by the Bodleian Library of Oxford University a bit more than 200 years ago. It is an extremely handsome volume, written in an elegant Greek script. The directors of the actual photography were Chet Grycz of Octavo and Richard Ovenden of the Bodleian, roughly dividing the work into photography and preservation. (This sort of work is a specialized art, largely developed within Octavo. Hans Hansen did the technical work.) This took place at Oxford in the fall of 2004. The output was a set of of 386 digital images at very high resolution, in essentially the same format as previous Octavo projects. CMI, the Bodleian Library, and Octavo now maintain copies of the original images for special purposes, but online copies at somewhat lower resolution are available at CMI and the website of the nonprofit organization *Libraries without Walls*, with whom Grycz now works.

The next two projects took place in Göttingen with the help of Yuri Tschinkel. Bernhard Riemann's famous 1859 manuscript "On the number of primes below a fixed bound", was photographed in 2005 by the Niedersächsische Staats- und Universitätsbibliothek Göttingen. This institution preserves many of the most famous mathematical manuscripts, including its best known treasure, Gauss' *Tagebuch*, which is stored in a special safe. Helmut Rohlfing, curator of manuscripts, directed the work. The images produced have been available since then on the CMI website.

Much larger in scope was the digitization of the Klein *Protokolle* at the Mathematisches Institut in Göttingen, described in this issue at length by Tschinkel and his colleague Eugene Chislenko. Technical work was here, too, the responsibility of Grycz and *Libraries without Walls*. The photographer was Ardon Bar Hama. He flew into Göttingen from Israel for three days of intense work, using a Leaf Aptus 75 camera with a digital back, capable of producing single images of 39 MB.

In all these projects as in similar ones involving the Internet, the copying of the works is only a small part of what must be done to make the works truly accessible. Chislenko is now working to edit and annotate the digitized volumes, and is engaged in research in the history of mathematics with this material as the primary source.

There is much more of value to be digitized in Göttingen, for long the home of many of the world's best known mathematicians, starting with Gauss and including more recently Hilbert and Siegel. The most recent CMI digitization project, currently under way with the Staats- und Universitätbibliothek, is the preservation of portions of Riemann's *Nachlass*.

Websites:
`http://www.claymath.org/library/historical`
`http://www.librarieswithoutwalls.org`
For an overview, and for the Klein *Protokolle* in particular
`http://www.librarieswithoutwalls.org/klein.html`.

—James Carlson, president, Clay Mathematics Institute.