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Measurement Uncertainty in Reverberation Chambers – I. Sample Statistics

Edition 2.0

L. R. Arnaut

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Abstract

We define and derive expressions for the intrinsic uncertainty associated with random fields inside a reverberation chamber. Uncertainties are derived based on exact sampling theory for sample sets generated by N independent stir states. A hierarchy of expressions and results is presented that correspond to increasing levels of complexity and accuracy of the statistical models employed. In particular, closed-form expressions for the boundaries of confidence intervals for ensemble statistics of the power and field magnitude are derived. The expressions become asymptotically exact for increasingly large N and are based on the sample average value and standard error only. For emission measurements, the standard deviation for the average total radiated power is shown to be proportional to $1/\sqrt{MN}$. By contrast, for the maximum received field in immunity measurements, the standard deviation is proportional to $1/[M^\beta \ln(N)]$ with parameter $\beta > 1/2$. The theoretical results are illustrated with measured data on an emitting electronic device and with measured maximum levels of field strength estimated from chamber validation data. The methods and results can be used in chamber validation, EMC testing (including emission, immunity, shielding, absorption, and susceptibility measurement), antenna measurements, and numerical simulations.

Keywords: measurement uncertainty, random electromagnetic vector fields, sampling distribution, confidence interval, hybrid stirring techniques.

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Executive Summary

We investigate the contribution of the randomness of the field to the overall measurement uncertainty (MU) in a mode-tuned or mode-stirred reverberation chamber (MT/MSRC). The random fluctuations of the field that cause this randomness arise as a result of mode stirring or scanning process and constitute the so-called intrinsic field uncertainty (IFU). The IFU is distinct from other contributions to MU such as chamber (site) imperfections, instrumentation uncertainty, etc.

For data sets consisting of N statistically independent sample values of the power or field strength evaluated during the tuning, stirring, and/or scanning process, we calculate the sample distribution, sample average, standard error, as well as the boundary locations and width of confidence intervals for the average value, the standard deviation, and the maximum or minimum values of the power (or energy density or field intensity), field magnitude (i.e., field strength), and complex-valued analytic electric or magnetic field. The ensemble statistics are expressed exclusively in terms of performance-based physical quantities, in order to aid in their practical evaluation from measured data. The analysis and results are limited here to ideal MT/MSRC operation, i.e., a chamber supporting a statistically isotropic, homogeneous, unpolarized, and incoherent interior field that yields chi-square and chi ensemble distributions for the power and field magnitude, respectively. These ensemble distributions can be associated with sample sets of infinite size ($N \rightarrow +\infty$) as a limiting case.

It is shown that different degrees of sophistication, elaboration, and precision for the statistical model of the sampled field can be introduced. In increasing order of accuracy, we distinguish between:

1. *Gauss normal approximation*: when N is large (typically $N > 30$), the sample average, standard deviation, maximum, and minimum values of the power or field strength exhibit probability distributions that can be closely approximated by a Gauss normal sampling distribution with N -dependent standard error. This approximation becomes asymptotically exact for $N \rightarrow +\infty$. The sample standard deviation then completely characterizes the IFU, and the expanded IFU can be determined, for example, with coverage factor $k = 1.960$ for a 95 % confidence interval. For easy practical use, we list simple analytic expressions for the boundaries of the asymptotic confidence intervals of the average value, standard deviation, and maximum value of the Cartesian and vectorial power and field strength;
2. *Asymptotic sampling distributions*: for finite N , particularly for relatively small values, the Student t sampling distribution provides a first-order correction to the Gauss normal approximation and results in wider uncertainty intervals. This distribution is useful when it is difficult to express the exact sampling distribution in closed form (for example, in hybrid stirring and/or

scanning techniques, convolutions of multiple measurands, combination of IFU with measurement instrumentation uncertainty (MIU), etc.), but when the resultant degrees of freedom of the field (represented by modal or spectral plane-wave components) is not asymptotically large. In particular, this method can be used for the sampling distribution of the maximum value in space-time sampling when the number of spatial measurement locations M is relatively small, provided still $N \gg 1$;

3. *Sampling distributions with a priori known or independently evaluated ensemble parameters:* these distributions are based on complete knowledge of the ensemble parameters used in the expression of the sampling distribution; typically, they assume ideal $\chi^{(2)}$ ensemble distributions for the power or field strength. The sampling distributions reduce to the aforementioned second or first cases when N becomes large or approaches infinity, respectively;
4. *Sampling distributions with parameters estimated from the sample set itself:* This case is the most general and, in practice, the most important one because the sample standard deviation is typically estimated from the measured data itself. It is shown that the thus estimated standard deviation itself exhibits relatively large uncertainties in case of relatively small sample sizes (typically of the same order of magnitude as the expected value, i.e., ensemble value of the standard deviation itself when $N < 800$), and diminishes only slowly with increasing sample size. This sampling uncertainty for the distribution parameters increases the IFU substantially, and must be properly accounted for. This case reduces to the previous one when the number of independent samples $\mathcal{N}$ used to estimate the distribution parameters increases to infinity (vanishing standard error of the estimated distribution parameters), *a fortiori* when $N \rightarrow +\infty$.

Compared to the asymptotic forms, the actual sample distributions are found to generally produce, in a first approximation, an upward shift of an $\eta\%$ confidence interval for the power or field strength for given N , most notably for $N < 30$. By contrast, the width of the confidence interval is much less unaffected.

The main results of IFU and its implications for the precision of the different types of EMC testing inside a MT/MSRC are as follows:

1. Relevant to radiated emissions, the IFU of the radiated power can be estimated and calculated from the measured reflection and transmission S-parameters for the empty (unloaded) chamber. For idealized condition of operation, the stir-averaged statistical impedance mismatch can be calculated from the stir-averaged insertion loss $\langle |S_{21}|^2 \rangle$ and has a significant effect on the un-

certainty of the transmitted power when $\langle |S_{21}|^2 \rangle$ is -7.8 dB or higher. For chamber validation, the mismatch becomes significant for $\langle |S_{21}|^2 \rangle \sim -10$ dB or higher. The standard error and the width of the confidence interval of the average emitted power are proportional to $1/\sqrt{N}$. The coefficient of variation (relative level of fluctuations) of the estimated average radiated power is of the order of $\sqrt{2/N}$ ($\sim [1.5 - 5 \log_{10}(N)]$ dB);

2. Relevant to chamber validation and immunity testing, the IFU for the received sample maximum field strength is estimated, based on ensemble statistics of the average value, standard deviation, and confidence intervals for the ensemble maximum field strength. The boundaries and widths of 95 % and 99 % confidence intervals for the maximum power and field strength are evaluated as a function of N and M . The width of a confidence interval for the sample average value of the maximum field strength or power is found to be proportional to $1/[M^\beta \ln(N)]$ with $\beta > 1/2$ for small M , and $1/[\sqrt{M} \ln(N)]$ for $M \rightarrow +\infty$. The confidence interval for the sample standard deviation of the maximum field strength decreases to zero for $M \rightarrow +\infty$ at a rate $1/\sqrt{M}$;
3. Relevant to chamber validation, the IFU for the normative field uniformity metric as specified in IEC 61000-4-21 Ed. 1 is evaluated for Cartesian field components at selected values of N . It is found that, for any value of N , the width of the uncertainty interval of the metric resulting from the relatively small values of M in the validation procedure is of comparable order of magnitude as the expected value of the metric (i.e., its value associated with $M \rightarrow +\infty$) itself. This is of particular importance for relatively small values of N (i.e., $N \ll 100$) and $M = 8$;
4. Relevant to testing and evaluation of sensitivity and susceptibility (e.g., fading in radio communication systems), the uncertainty of the minimum value of the power or field strength demonstrate a comparatively weak dependence on the length of the stir sequences. It is shown that for hybrid stirring, the statistics of minimum field strength or power now depend on N and M in a more complicated manner than via their product, unlike for the maximum value;
5. Relevant to emission, immunity, susceptibility, and shielding measurements, a general characteristic of EMC/EMI testing in a MTRC is that the contribution of the IFU to the MU is only limited by the number of independent stir states N that are generated, i.e., the IFU becomes smaller the *greater* the randomness of the field (in the statistical sense) and can be made arbitrarily small (in principle) provided N can be increased without bound;
6. With a view to further reducing the contribution of the IFU to MU, the application of a hybrid (multiple) stirring method or additional spatial scanning producing $M-1$ additional sample sets

(stir sweeps) is advocated, discussed, analyzed, and found to reduce the IFU by a further factor $1/\sqrt{M}$. For this method, the effective number of statistically independent sample values is estimated from the number of independent stir or sample states associated with the individual stirring processes.

In practice, the maximum number of independent stir states $N_{\max}$ (i.e., the size of the “universe” from which samples are taken) that a given chamber can produce is more or less limited, even for continuous mode stirring, particularly at relatively low frequencies of operation. Results based on a theoretically unlimited maximum number of independent samples ($N_{\max} \rightarrow +\infty$) are compared with those for a finite value ($N_{\max} \not\rightarrow +\infty$). It is found that the reduction of this ensemble size gives rise to a substantial reduction of the IFU when N approaches $N_{\max}$. An equivalent number of stir states N' is derived that enables extending certain results on MU for an infinite universe to a finite one.

Experimental results are presented for the estimated IFU for radiated emissions, calculated for an actual digital electronic device (built around a comb generator) that was measured at NPL for frequencies from 100 MHz to 1300 MHz. 95 % confidence intervals for the values of the Cartesian and total radiated power are confirmed to be of the order of 1.5 to 2 dB in this frequency range, and can be further decreased by increasing the number of spatial measurement positions (viz., locations or orientations of the equipment under test (EUT)) M , by scaling by a factor that asymptotically approaches $\sqrt{2/M}$ for large M . Chamber validation data are used to estimate statistics of the average value of the Cartesian power and field strength and their maximum-to-average ratio, together with their asymptotic 95 % confidence intervals.

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Preface to Edition 2.0

Compared to the first edition of this report (May 2008), Sec. 4 has been revised, providing more general expressions for impedance mismatch at the transmitter and receiver sides (including source and detector mismatch) and incorporating the effect of correlations between S-parameters on the estimation of uncertainty for emissions measurements. Sec. 5 contains further analysis that allows for improved characterization of immunity for small sample sizes. A number of typographical errors in text, formulae and figures have been corrected. Some textual changes have been made for improved clarity.

December 2008

List of Acronyms

ACF	autocorrelation function
AGC	automatic gain control
AWGN	additive white Gaussian noise
(c)df	(cumulative) distribution function
CLF	chamber loading factor
CVF	chamber validation factor
CW	continuous wave
EM	electromagnetic
EMC	electromagnetic compatibility
EME	electromagnetic environment
EUT	equipment under test
FAC	fully anechoic chamber
GTEM	gigahertz transverse electromagnetic
IEC	International Electrotechnical Committee
IFU	intrinsic field uncertainty
LUF	lowest usable frequency
MIU	measurement instrumentation uncertainty
MT/MSRC	mode-tuned or mode-stirred reverberation chamber
MU	measurement uncertainty
OATS	open-area test site
pdf	probability density function
SAC	semi-anechoic chamber
TEM	transverse electromagnetic

1 Introduction

The evaluation of measurement uncertainty (MU) and numerical accuracy is fundamental to good practice in metrology and in mathematical modelling or simulation. This includes their specific application to electromagnetic compatibility (EMC), characterization of antennas and complex electromagnetic environments (EMEs), computational electromagnetics, etc. To quantify MU, the lower and upper bounds (limits) and/or the width of an $\eta\%$ confidence interval of a physical quantity of interest or a function thereof, for specified confidence level η , are of central importance. Such an interval expresses the range of experimental values that the measurand is expected to take with probability $\eta/100$.

Conventional EMC test and measurement facilities (e.g., OATS, FAC, SAC, (G)TEM cells, etc.) operate on the basis of *plane-wave excitation and/or propagation*. In anechoic or semi-anechoic EMEs, the spatial field structure is deterministic (i.e., fixed) and regular (i.e., “simple”), in the sense that the idealized structure is the same as in unbounded (infinite) free space, possibly including the effect of a ground plane and a small number of isolated weak scatterers. Under sinusoidal (continuous wave, CW) excitation, the field structure – as governed by the continuity conditions of classical EM fields for stationary boundaries – exhibits a spatially and temporally harmonic dependence. In such plane-wave EMEs, any MU originates from random fluctuations (Type-A evaluation of uncertainties) by external factors occurring within a set of measured values performed under nominally identical conditions of the EME in space-time-frequency. For deterministic EMEs, the largest contribution to the budget of the MU [5] is made by the imperfections of the site, deviating from ideal reflection-free conditions.

By contrast, an alternative measurement facility is the mode-tuned or mode-stirred reverberation chamber (MT/MSRC) [2] representing a *non-plane-wave*¹ EME, which generates an ensemble (universe, population) of different but statistically equivalent field patterns and field values in space and time (so-called stir states). The characteristics of a MT/MSRC are based on its extreme sensitivity of the interior resonant field to changes in geometry (boundary configuration) and/or different EM excitation conditions (frequency, phase, bandwidth, polarization, etc.) inside an overmoded cavity, whence the field can be modelled as a quasi-random vector function of location and time. In fact, a MT/MSRC is a random field generator that *relies* on a significantly large number of stir states to be realized. Typically, but not exclusively, this number is chosen to be large in order to provide results with sufficiently low IFU. Measurements at any *single* location and state of a MT/MSRC exhibit a

¹Owing to the overmoded regime, the local electric and magnetic fields are quasi-independent quantities and their directions of polarization are not necessarily mutually orthogonal. The former is a consequence of the quasi-random nature of the field; the latter follows by noting the non-orthogonality of the superposition of two plane-wave fields, each with mutually orthogonal electric and magnetic fields with arbitrary field magnitudes and specific wave impedances.

substantial *intrinsic* (inherent) statistical² field uncertainty (IFU), even under ideal conditions. The particular value of the field by itself is usually not meaningful, because of the large uncertainties associated with a single measured value. This is often expressed by the statement that an EMC test inside a MT/MSRC is inherently a statistical test, signifying that it exploits the inherent statistical (stochastic) nature of the field generation mechanism to provide results on the expected value, confidence interval, etc., for the ensemble.

Except for exceedingly high Q-factors of a MT/MSRC at the frequency of operation, and/or except when a very large number of independent stir states is generated through multiple stirring mechanisms applied in succession (so-called *hybrid stirring* [3]), the measurement instrumentation uncertainty (MIU) in a MT/MSRC is usually at least one order of magnitude smaller than the IFU of the random field itself. Therefore, MIU can usually be neglected. General aspects of the evaluation of MU in EMC and ICT, including sources of MU and specific aspects of MIU, are listed and described in [4]– [8]. If the MIU and other sources of uncertainty or error have a magnitude that is comparable to that of the (nonreduceable) IFU associated with the physical randomness of the field, then the classical method of combining different sources of uncertainty in the evaluation of a combined uncertainty comes into force: the IFU constitutes simply one other (but major) contribution in the uncertainty budget. The evaluation of other contributions to the MIU in a MTRC are the subject of recent and ongoing investigations, cf., e.g., [9, Sec. 4.8].

2 Outline

In this report, the focus is on the IFU of a MT/MSRC, i.e., on contributions to the combined MU that are directly related to the random nature of the field itself. The analysis applies to a single reference equipment under test (EUT) whose EM radiated emission, immunity or susceptibility is evaluated. Since a MT/MSRC test involves generating a statistical set of values of the local field, the fundamental EM quantity is power or, equivalently, energy density or field intensity³, which are square-law scalar functions of the complex field, rather than the complex-valued field itself or its (real) magnitude⁴. We provide an analysis that is based on the determination of the average value and standard deviation of

²The term “statistical” is explicitly added here, to distinguish it from the notion of intrinsic uncertainty, defined (with a different meaning) in [4] for plane-wave EMC test facilities.

³In this report, the electric power is considered to be normalized by the wave impedance, i.e., representing the field intensity: $P\eta_0 = |E|^2$. All results for the actual power follow from scaling the distribution for the intensity.

⁴Since a MT/MSRC is a non-plane wave EME, the conversion between field magnitude and power is not trivial. For some of their statistics (e.g., the maximum value, obtained at a certain stir state), a conversion may not even be possible in an precise manner. The issue is further discussed in Sec. 4.3.

the radiated (emitted) or incident power or the field strength. In this first of two Parts, the analysis is primarily based on statistics, i.e., first- and second-order sample-statistical moments (average value and standard deviation) and confidence intervals. Such a restricted characterization is usually sufficient for EMC applications, where the IFU is typically large compared to the other MU contributions, including MIU (source, receiver, cables, connectors, etc.). In applications characterized by smaller MU (e.g., antenna characterization), where a more precise determination of confidence intervals is required, a more in-depth analysis based on the pertinent statistical distribution functions is needed. This will be the subject of the analysis in Part II.

The report is organized as follows. In Sec. 3, some results from asymptotic and exact sampling theory are reviewed and applied or extended to the case of Cartesian or vectorial power of field strength as the measurand. The focus is on the sample average, sample standard deviation, $\eta\%$ sample confidence intervals (location and width), and extreme value statistics. Both ideal and realistic chambers are considered. In Sec. 4, the IFU specifically associated with chamber validation and emission measurements in a MT/MSRC is investigated, with focus on the effect of statistical impedance mismatch on the MU (standard error) of the estimated emitted power. Again, both ideal and realistic chambers are considered. Closed-form expressions are given for asymptotic confidence intervals for the average value and standard deviation of the emitted power and received power inside an ideal chamber. The theory is applied to experimental results for an actual radiating EUT, providing realistic values of confidence intervals associated with the average radiated power that can be expected in practice. In Sec. 5, IFU for immunity testing is considered, with the focus on confidence intervals for the maximum received field strength or power. Closed-form expressions for the standard error and confidence intervals for the ensemble and sample-statistical maximum Cartesian field strength or power are obtained. The effect of hybrid stirring on the reduction of the IFU is investigated. Asymptotically exact sampling distributions for the maximum field strength or power, for both single and hybrid stirring and for Cartesian and vector fields, are derived. Closed-form expressions are given for the asymptotic confidence intervals for the average value and standard deviation of the maximum field strength or power inside an ideal chamber, for practical use. In Sec. 6, the corresponding sample statistics of the minimum received power or field strength are considered. Finally, in Sec. 7, the main results are summarized and reviewed.

Only idealized conditions of operation (overmoded regime) allow for relatively simple expressions for the evaluation of MU in a MT/MSRC. The treatment here is limited to ideal Gauss normal and chi(-squared) parent distributions for the ensemble field. The expression of IFU for undermoded MT/MSRC and for boundary fields is more complicated, but can in principle be carried out using the

same results or techniques described in this report.

Regarding notations used in this report, $\text{Avg}(X)$, $\text{Std}(X)$ (or S_X), and $\text{Var}(X)$ (or S_X^2) on the one hand, and $\text{avg}(X)$, $\text{std}(X)$ (or s_X), and $\text{var}(X)$ (or s_X^2) on the other, represent random variables and associated values of the sample average, sample standard deviation, and sample variance of the random variable X , respectively. A sampling probability density function (pdf) of the ensemble pdf $f_X(x)$ with $2p$ degrees of freedom⁵ that is obtained following hierarchical sampling of $f_X(x)$ using M sets of N samples each, with replacement, is denoted by $f_X(x; N, M)$, i.e., with parameters M and N . In some cases, we dispense with the convention of using upper-case letters to denote random variables and corresponding lower-case symbols for their sample values, because of the large number of different symbols required and for historic reasons. For example, N , M , Q , A_e refer to deterministic EM quantities. Other notations will be self-explanatory. Quantiles and percentiles are represented by q and $\eta\%$, respectively. A frequentist's viewpoint is adopted, leading to the expression of MU in terms of random errors [10, Sec. 3.2.2] which are directly related to the physical origin of the resulting uncertainty.

The analysis applies to mode-tuned and quasi-stationary mode-stirred operation of reverberation chambers. For brevity, we shall employ the notion of mode stirring as a collective for mode stirring or mode tuning, and to *stir states* to refer to different cavity realizations (e.g., obtained using different positions of the paddle wheel, source or receiver positions, etc.) generated in either mode stirring or mode tuning.

3 Sample Statistics, Standard Errors, and Sampling Distributions of Received Power and Field Strength

3.1 Ideal Chamber ($N_{\max} \rightarrow +\infty$)

In the idealized case of arbitrarily high frequency and maximum efficiency of the stirring mechanism(s), mode stirring is capable of generating an unlimited number ($N_{\max}$) of statistically independent sample values of the received power, P_R , i.e., $N_{\max} \rightarrow +\infty$. Such an infinite-sized set of values defines the ensemble statistics of the power and constitutes a “pool” of values for generating sampled values. Of course, in practice, the effort expended on collecting such data has to be limited. This results in a finite-sized subsample of N statistically independent values being generated.

Sample statistics for samples of size N can be obtained using exact sampling theory; cf. e.g., [11,

⁵The number of degrees of freedom is always an even number, because power, intensity, and magnitude involve (the sum of squares of) two field components in quadrature (real and imaginary parts of a complex variable).

Chs. 27–29], [12, Chs. 10–12]. Each sample set consisting of N sample data points is then used to calculate one number (sample statistic) from these points, e.g., the sample mean, sample standard deviation, etc. Since the sample size is finite, the value of this sample statistic exhibits fluctuations with each new realized sample set of size N . Exact sampling theory allows for evaluation of the uncertainty associated with this calculated number when estimating the corresponding ensemble characteristic. The theory is applied here to χ^2 parent distributions with two or six degrees of freedom ($p = 1$ or 3), depending on whether the electric or magnetic power associated with a Cartesian component of a random field or the full 3-D vector random field, respectively, is of interest.

It is emphasized that for the analysis and results in this Section, we only require the condition $N_{\max} \rightarrow +\infty$, whereas the sample size N is arbitrary and does not need to be large. Background information and results on ensemble statistics for the χ^2_p parent distributions and their applications to MT/MSRCs can be found, e.g., in [3], [11]–[16].

3.1.1 Mean Value

The ensemble average $\langle X \rangle$ and ensemble standard deviation σ_X can be estimated from the sample average $\text{Avg}(X)$ (which, as mentioned before, is itself a random variable with respect to different sample sets) and the associated sample standard deviation $s_{\text{Avg}(X)}$ (also known as the standard error) of $\text{Avg}(X)$.

3.1.1.1 Power The sample average received power is

$$\text{avg}(P_R) \triangleq \frac{\sum_{\ell=1}^N P_R(\ell)}{N} \quad (1)$$

in which $P_R(\ell)$ is the received power measured at stir state ℓ and the (arithmetic) averaging involves summing over all N elements within one particular sample set of N generated stir states. An estimation of the corresponding ensemble (population) parameter, i.e., the expected value $\langle P_R \rangle \triangleq \int_0^{+\infty} p_R f_{P_R}(p_R) dp_R \simeq \sum_{\ell=1}^N P_R(\ell) f_{P_R}(p_R = P_R(\ell))$, follows as

$$\langle P_R \rangle = \langle \text{Avg}(P_R) \rangle \simeq \text{avg}(P_R) \quad (2)$$

where the approximation becomes increasingly accurate for $N \rightarrow +\infty$ in (1). In (2), $\text{Avg}(P_R)$ is the induced random variable associated with $\text{avg}(P_R)$ corresponding with the values of $\text{avg}(P_R)$ generated by different but statistically equivalent sample sets of equal size N , and $\langle \cdot \rangle$ represents averaging with respect to the ensemble of all possible sample sets of size N (taken from an infinite-sized population).

For the estimation of $\langle P_R \rangle$, the N stir states do not need to generate statistically independent⁶ values $P_R(\ell)$. However, for the standard deviation⁷, distribution function, and confidence interval of $\text{Avg}(P_R)$, the independence of sampled values is required, as we shall assume from here onward.

The sample standard deviation (also known as the standard error) of $\text{Avg}(P_R)$ is⁸

$$s_{\text{Avg}(P_R)} = \frac{s_{P_R}}{\sqrt{N}} = \sqrt{\frac{\sum_{\ell=1}^N [P_R(\ell) - \text{avg}(P_R)]^2}{N(N-1)}} \simeq \sqrt{\frac{\sum_{\ell=1}^N [P_R(\ell) - \langle P_R \rangle]^2}{N^2}} \simeq \frac{\sigma_{P_R}}{\sqrt{N}} \quad (3)$$

where the ensemble standard deviation is $\sigma_{P_R} \triangleq \sqrt{\langle P_R^2 \rangle - \langle P_R \rangle^2}$ and the sample standard deviation of P_R for a set of N samples is defined as⁹ [10, Sec. 4.2.2]

$$s_{P_R} \triangleq \sqrt{\frac{\sum_{\ell=1}^N [P_R(\ell) - \text{avg}(P_R)]^2}{N-1}} \quad (4)$$

where the denominator $N-1$ represents the number of degrees of freedom in a sample of size N . When averaging over the ensemble of all sample sets of size N , $s_{\text{Avg}(P_R)}$ converges to the sampling standard deviation of $\text{Avg}(P_R)$, given by

$$\sigma_{\text{Avg}(P_R)} = \frac{\sigma_{P_R}}{\sqrt{N}}. \quad (5)$$

An estimation of the ensemble standard deviation, σ_{P_R} , which corresponds to $N \rightarrow +\infty$ in (3), is

$$\sigma_{P_R} \simeq s_{P_R} \quad (6)$$

For large values of N (typically¹⁰, $N > 30$ in the calculation of $\text{Avg}(P_R)$), the values of $\text{Avg}(P_R)$

⁶A generalized mean value, accounting for the statistical dependence (provided it can be quantified), can also be used. Several procedures exist for extracting the number of approximately uncorrelated samples, which are often a good approximation for statistical independence.

⁷Alternative measures of fluctuation based on consecutive sample differences exist that do not require the samples to be independent [17], [18]. In Sec. 4.1.2.4, we shall give a first-order correction of s_{P_R} $s_{\text{Avg}(P_R)}$ for the case where P_R or $\text{Avg}(P_R)$ can be expressed as a functional form involving correlated quantities.

⁸The last approximation in (3) becomes exact when additional averaging is performed with respect to all possible sample sets of size N from the given data set.

⁹In (4), the denominator $N-1$ accounts for the fact that $s_{P_R}^2$ for a sample of size N and given $\text{avg}(P_R)$ possesses only $N-1$ degrees of freedom. This yields an unbiased estimate for the sample *variance*, $s_{P_R}^2$. Alternatively, $s_{P_R}^2$ could have been defined with N instead of $N-1$ in the denominator of (4), is obtained from a maximum-likelihood estimation of the sample variance in the case where the ensemble distribution is Gauss normal. This is sometimes [19] taken as the definition for the ensemble standard deviation, i.e., $\sigma_{P_R} \triangleq \sqrt{(N-1)/N} s_{P_R}$. Generally, for the sample standard deviation s , the corresponding factor $N-\frac{3}{2}$ produces a nearly unbiased estimate in case of a Gauss normal distribution, to first order in $1/N$. An exact correction factor for s will be given in Sec. 3.1.2.

¹⁰The choice $N = 30$ as a threshold value corresponds to an effective number of degrees of freedom plus one for a Student t sampling distribution for the Gauss normal ensemble distribution that results in a relative value $k/k_\infty \simeq 2.042/1.960 \simeq 1.042$ for a 95% coverage factor (cf. Table 6 in Appendix B), i.e., a difference in width of less than 5% compared to the asymptotic Gauss normal coverage. By way of comparison, corresponding values for $N = 10, 40$, and 100 are $2.23/1.96 \simeq 1.14$, $2.021/1.960 \simeq 1.031$, and $1.984/1.960 \simeq 1.012$, respectively.

are approximately Gauss normally distributed¹¹, on account of the central limit theorem, with a 95% confidence interval for $\langle \text{Avg}(P_R) \rangle$ given by¹²

$$\left[\text{avg}(P_R) - 1.960 \, s_{\text{Avg}(P_R)}, \text{avg}(P_R) + 1.960 \, s_{\text{Avg}(P_R)} \right]. \quad (7)$$

Since $\sigma_{P_{R,\alpha}} = \langle P_{R,\alpha} \rangle$ for a χ_2^2 distributed Cartesian $P_{R,\alpha}$ ($\alpha = x, y, z$), the ensemble value of the mean-normalized width of this interval follows from (2) and (3) as $3.920/\sqrt{N}$, i.e., $[5.93 - 5 \log_{10}(N)]$ dB. For the χ_6^2 distributed vectorial (total) $P_{R,t}$ for which $\sigma_{P_{R,t}} = \langle P_{R,t} \rangle/\sqrt{3}$, the relative width is narrower, viz., $2.263/\sqrt{N}$, i.e., $[3.54 - 5 \log_{10}(N)]$ dB.

For arbitrary (including relatively small) values of N , the interval (7) is only an approximation for an¹³ actual 95 % confidence interval. A precise calculation of the latter requires knowledge of the distribution of $\text{Avg}(P_R)$ for N statistically independent and identically distributed (i.i.d.) samples $P_R(\ell)$ with a χ^2 parent distribution. This distribution is obtained as follows. The pdf $f_X(x)$ and cumulative distribution function (cdf) $F_X(x)$ of the χ_{2p}^2 distributed $X \triangleq P_R \propto |E_R|^2$ – being the sum of squares of $2p$ real (or, equivalently, p circular complex) Gauss normal distributed field components – are given, in self-sufficient form [21], [22], by

$$f_X(x) = \frac{p^{p/2}}{\Gamma(p) \sigma_X} \left(\frac{x}{\sigma_X} \right)^{p-1} \exp \left(-\sqrt{p} \frac{x}{\sigma_X} \right), \quad F_X(x) = \frac{\gamma \left(p, \sqrt{p} \frac{x}{\sigma_X} \right)}{\Gamma(p)} \quad (8)$$

where $p = 1$ or 3 for the Cartesian or vectorial power (or field intensity), respectively, with¹⁴

$$\gamma(p, u) \triangleq \int_0^u t^{p-1} \exp(-t) dt, \quad \Gamma(p) \triangleq \int_0^{+\infty} t^{p-1} \exp(-t) dt \quad (9)$$

defining incomplete and complete gamma functions, respectively [25, Secs. 8.31 and 8.35]. The mean and standard deviation are

$$\langle X \rangle = p \, \sigma_{E_{R,\alpha}}^2 = 2p \, \sigma_{E'_{R,\alpha}}^2 = \sqrt{p} \, \sigma_X, \quad \sigma_X = 2\sqrt{p} \, \sigma_{E'_{R,\alpha}}^2 \quad (10)$$

¹¹The same holds true for the sample average of the complex analytic (quasi-harmonic) field and its magnitude.

¹²The confidence coefficient or coverage factor 1.960 in (7) is an approximation for 1.959964..., valid for $N \rightarrow +\infty$. For different confidence levels and different distributions, the applicable coefficient changes accordingly. Rounding this value to 2 results in a marginally higher confidence level (viz., 95.45 %) in case of a Gauss normal distribution [10], [13, Sec. 1.10]. Here, the confidence level, e.g., 95 % is the primary quantity interest, and the confidence coefficient (whichever distribution it is applied to) is the derived quantity.

¹³For a given confidence coefficient, the confidence interval can be defined in different ways, e.g., equiprobabilistic limits of the interval, shortest or longest interval, etc. In this report, we use the equiprobabilistic definition.

¹⁴NB: some numerical implementations of $\gamma(p, u)$ (e.g., MATLAB) employ instead the normalized definition $\gamma(p, u) \triangleq [1/\Gamma(p)] \int_0^u t^{p-1} \exp(-t) dt$, with which the denominator of $F_X(x)$ in (8) vanishes.

where $\sigma_{E'_{R,\alpha}^{(\iota)}}$ is the standard deviation of the real or imaginary part¹⁵ of the interior¹⁶ Cartesian electric field $E_{R,\alpha}$ in its Gabor analytic representation. The fluctuations of this idealized statistically incoherent field (allowing for the summation of powers because of vanishing of cross-product contributions) define a circular Gauss normal quasi-harmonic random process. Because of the additivity property of the χ^2 distribution, the sum $Y \triangleq \sum_{\ell=1}^N P_R(\ell) = \sum_{\ell=1}^{pN} P_{R,\alpha}(\ell) \propto \sum_{\ell=1}^{pN} \{[E'_{R,\alpha}(\ell)]^2 + [E''_{R,\alpha}(\ell)]^2\}$ has a χ_{2pN}^2 sampling pdf, given by

$$f_Y(y; N) = \frac{(pN)^{pN/2}}{\Gamma(pN) \sigma_Y} \left(\frac{y}{\sigma_Y} \right)^{pN-1} \exp \left(-\frac{\sqrt{pN} y}{\sigma_Y} \right) \quad (11)$$

with

$$\langle Y \rangle = 2pN \sigma_{E'_{R,\alpha}^{(\iota)}}^2, \quad \sigma_Y = 2\sqrt{pN} \sigma_{E'_{R,\alpha}^{(\iota)}}^2. \quad (12)$$

Upon normalizing Y by $1/N$, the sampling pdf of $\text{Avg}(P_R) \equiv Y/N$ thus follows as a re-scaled χ_{2pN}^2 pdf with $\langle Y \rangle = N \langle \text{Avg}(P_R) \rangle$ and $\sigma_Y = \sqrt{N} \sigma_X = N \sigma_{\text{Avg}(X)}$, viz.,

$$f_{\text{Avg}(P_R)}[\text{avg}(P_R); N] = N f_Y[y = N \text{avg}(P_R); N] \quad (13)$$

$$= \frac{(pN)^{pN/2}}{\Gamma(pN) (N \sigma_{\text{Avg}(P_R)})^{pN}} [N \text{avg}(P_R)]^{pN-1} \exp \left(-\frac{N \sqrt{pN} \text{avg}(P_R)}{N \sigma_{\text{Avg}(P_R)}} \right) \quad (14)$$

$$= \frac{(pN)^{pN/2}}{\Gamma(pN) \sigma_{\text{Avg}(P_R)}^{pN}} \left(\frac{\text{avg}(P_R)}{\sigma_{\text{Avg}(P_R)}} \right)^{pN-1} \exp \left(-\sqrt{pN} \frac{\text{avg}(P_R)}{\sigma_{\text{Avg}(P_R)}} \right) \quad (15)$$

with sampling mean and sampling standard deviation of $\text{Avg}(P_R)$ given by

$$\langle \text{Avg}(P_R) \rangle = p \sigma_{E_{R,\alpha}}^2 = 2p \sigma_{E'_{R,\alpha}^{(\iota)}}^2 = \sqrt{pN} \sigma_{\text{Avg}(P_R)}, \quad \sigma_{\text{Avg}(P_R)} = 2\sqrt{\frac{p}{N}} \sigma_{E'_{R,\alpha}^{(\iota)}}^2. \quad (16)$$

The form (15) assumes the ensemble standard deviation $\sigma_{\text{Avg}(P_R)}$ (and, hence, σ_{P_R} or σ_{E_R}) to be known exactly. In practice, this parameter is typically estimated from the set of N sampled values itself. In this case, the sample standard deviation s_{P_R} or s_{E_R} exhibits increased uncertainty and gives rise to sampling F -distributions for P_R and $\text{Avg}(P_R)$, as detailed in Appendix A.

Figure 1(a) shows mean-normalized boundaries $\xi_{(1 \pm \eta/100)/2}^{\pm} / \langle \text{Avg}(P_R) \rangle$ of an $\eta\%$ confidence interval for χ_{2pN}^2 distributed $\text{Avg}(P_{R,\alpha})$ (i.e., for $p = 1$) with $\eta = 95, 99$, and 99.5 . The calculation is based

¹⁵The standard deviation of the real or imaginary part of the field should not be confused with the standard deviation of the complex field itself. With to the extended definition (99) for the variance of complex-valued variates, $\sigma_{E_{R,\alpha}} = \sqrt{2} \sigma_{E'_{R,\alpha}^{(\iota)}}$ because $\langle (E'_{R,\alpha}^{(\iota)})^4 \rangle = 3 \langle (E'_{R,\alpha}^{(\iota)})^2 \rangle^2$, whereas $\langle |E|^4 \rangle = 2 \langle |E|^2 \rangle^2$ for unbiased circular E .

¹⁶Results for boundary electric or magnetic fields at the interface with a perfect electric conducting (PEC) half-space are obtained for $p = 1$ and 2 , respectively, or vice versa for a perfect magnetic conducting (PMC) half-space [23], [24]. For finite distances ($0 < kd < +\infty$) and/or dielectric or magnetic half-spaces (rather than PEC or PMC interfaces), closed-form expressions for the ensemble distributions of the fields can still be obtained [23], [24] and their sample distributions can be derived by simple extension of the results in this Section.

on the inversion of the cdf¹⁷ of $\text{Avg}(P_R)$, i.e., as the two solutions of

$$F_{\text{Avg}(P_R)}[\text{avg}(P_R) = \xi_{(1 \pm \eta/100)/2}^\pm; N] \equiv \frac{\gamma\left(pN, \sqrt{pN} \frac{\xi_{(1 \pm \eta/100)/2}^\pm}{\sigma_{\text{Avg}(P_R)}}\right)}{\Gamma(pN)} = \frac{1 \pm \eta/100}{2} \quad (17)$$

for real positive (but not necessarily integer) values of N . The actual 95 % interval is compared with the approximate interval (7) in the same Figure, and similarly for $\eta = 99$ and 99.5. When $N \rightarrow +\infty$, the sample average is seen to approach the ensemble expected value, as a manifestation of the law of large numbers. Their difference tends to zero, in the mean, at a rate $1/\sqrt{N}$.

Figure 1(b) shows the corresponding widths of the interval, defined as the mean-normalized difference of the upper and lower percentiles, i.e.,

$$\frac{\xi_{(1+\eta/100)/2}^+ - \xi_{(1-\eta/100)/2}^-}{\langle \text{Avg}(P_{R,\alpha}) \rangle}. \quad (18)$$

The graphs show that the asymptotic Gaussian approximation (i.e., valid for $pN \rightarrow +\infty$) predicts the *width* of the confidence interval of $\text{Avg}(P_{R,\alpha})$ quite accurately, even for small values of N , whereas the *location* of its upper boundary is considerably underestimated for small values of N , compared to the actual boundary for χ_{2pN}^2 .

Confidence interval boundaries and widths for the vector (total) power $\text{Avg}(P_{R,t})$ (i.e., for $p = 3$) are shown in Fig. 2. The interval widths for $\text{Avg}(P_{R,t})$ are smaller than those for $\text{Avg}(P_{R,\alpha})$ by a factor $1/\sqrt{3}$ (~ -2.4 dB). This is a general feature of χ^2 pdfs: the larger the number of degrees of freedom, the smaller the mean-normalized uncertainty.

3.1.1.2 Field Amplitude In a similar way, the sample average of the received electric field amplitude $\text{Avg}(|E_R|)$ exhibits a χ_{2pN} distribution, as the root of the sum of squares of N i.i.d. sample field magnitudes $|E_R(\ell)|$, all having identical χ_{2p} ensemble distributions if the governing stochastic process is wide-sense stationary. The sampling pdf of $\text{Avg}(|E_R|)$ follows with the aid of the pdf of $Y \triangleq \sqrt{\sum_{\ell=1}^N |E_R(\ell)|^2}$. Based on the χ_{2p} pdf and cdf of $X \triangleq |E_R|$ [21], i.e.,

$$f_X(x) = \frac{2 \left[p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2 \right]^p}{\Gamma(p) \sigma_X} \left(\frac{x}{\sigma_X} \right)^{2p-1} \exp \left\{ - \left[p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2 \right] \left(\frac{x}{\sigma_X} \right)^2 \right\} \quad (19)$$

¹⁷For integer N , a χ^2 distribution for $\text{Avg}(P_R)$ holds, in which case its pdf (15) and cdf (17) can be expressed in terms of a finite series of products of polynomial and exponential functions of $\text{avg}(P_R)/\sigma_{P_R}$. In practical applications (typically, in oversampled MT/MSRCs), N is often estimated, in general, as a non-integer *equivalent* number of statistically independent states, in which case (17) is generally applicable. The situation is similar to the case of the χ_{2p}^2 parent distributions themselves, which generalize to gamma distributions in the case of non-integer real values of p for correlated field components.

and

$$F_X(x) = \frac{\gamma\left(p, \left[p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)}\right)^2\right] \left(\frac{x}{\sigma_X}\right)^2\right)}{\Gamma(p)} \quad (20)$$

respectively, with

$$\langle X \rangle = \sqrt{2} \frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \sigma_{E'_{R,\alpha}}, \quad \sigma_X = \sqrt{2 \left[p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)}\right)^2 \right] \sigma_{E'_{R,\alpha}}^2}, \quad (21)$$

the sampling pdf of Y is

$$f_Y(y; N) = \frac{2 \left[pN - \left(\frac{\Gamma(pN+\frac{1}{2})}{\Gamma(pN)}\right)^2 \right]^{pN}}{\Gamma(pN) \sigma_Y} \left(\frac{y}{\sigma_Y}\right)^{2pN-1} \exp \left\{ - \left[pN - \left(\frac{\Gamma(pN+\frac{1}{2})}{\Gamma(pN)}\right)^2 \right] \left(\frac{y}{\sigma_Y}\right)^2 \right\} \quad (22)$$

with¹⁸

$$\langle Y \rangle = \sqrt{2} \frac{\Gamma(pN+\frac{1}{2})}{\Gamma(pN)} \sigma_{E'_{R,\alpha}}, \quad \sigma_Y = \sqrt{2 \left[pN - \left(\frac{\Gamma(pN+\frac{1}{2})}{\Gamma(pN)}\right)^2 \right] \sigma_{E'_{R,\alpha}}^2}. \quad (24)$$

With $\langle Y \rangle = N \langle \text{Avg}(|E_R|) \rangle$ and $\sigma_Y = \sqrt{N \sigma_{|E_R|}^2} = N \sigma_{\text{Avg}(|E_R|)}$, the sampling pdf of $\text{Avg}(|E_R|)$ is obtained as $f_{\text{Avg}(|E_R|)}[\text{avg}(|E_R|); N] = N f_Y[y = N \text{avg}(|E_R|); N]$, i.e., as a re-scaled χ_{2pN} pdf:

$$\begin{aligned} f_{\text{Avg}(|E_R|)}[\text{avg}(|E_R|); N] &= \frac{2 \left[pN - \left(\frac{\Gamma(pN+\frac{1}{2})}{\Gamma(pN)}\right)^2 \right]^{pN}}{\Gamma(pN) \sigma_{\text{Avg}(|E_R|)}} \left(\frac{\text{avg}(|E_R|)}{\sigma_{\text{Avg}(|E_R|)}} \right)^{2pN-1} \\ &\times \exp \left\{ - \left[pN - \left(\frac{\Gamma(pN+\frac{1}{2})}{\Gamma(pN)}\right)^2 \right] \left(\frac{\text{avg}(|E_R|)}{\sigma_{\text{Avg}(|E_R|)}} \right)^2 \right\} \end{aligned} \quad (25)$$

with sampling mean and sampling standard deviation of $\text{Avg}(|E_R|)$ given by

$$\langle \text{Avg}(|E_R|) \rangle = \sqrt{2} \frac{\Gamma(pN+\frac{1}{2})}{\Gamma(pN)} \sigma_{E'_{R,\alpha}}, \quad \sigma_{\text{Avg}(|E_R|)} = \sqrt{\frac{2}{N} \left[pN - \left(\frac{\Gamma(pN+\frac{1}{2})}{\Gamma(pN)}\right)^2 \right] \sigma_{E'_{R,\alpha}}^2}. \quad (26)$$

Percentiles $\xi_{(1\pm\eta/100)/2}^\pm$ and confidence intervals of $\text{Avg}(|E_R|)$ follow from the inversion of the cdf as

$$F_{\text{Avg}(|E_R|)}[\text{avg}(|E_R|) = \xi_{(1\pm\eta/100)/2}^\pm; N] \equiv \frac{\gamma\left(pN, \left[pN - \left(\frac{\Gamma(pN+\frac{1}{2})}{\Gamma(pN)}\right)^2\right] \times \left(\frac{\xi_{(1\pm\eta/100)/2}^\pm}{\sigma_{\text{Avg}(|E_R|)}}\right)^2\right)}{\Gamma(pN)} = \frac{1 \pm \eta/100}{2} \quad (27)$$

¹⁸For the numerical calculation of the distribution and statistics of $\text{Avg}(|E_R|)$ for $pN \gg 1$, the following asymptotic series expansion is useful [26, Problem 9.60]:

$$\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x)} = \sqrt{x} \left(1 - \frac{1}{8x} + \frac{1}{128x^2} + \frac{5}{1024x^3} - \frac{21}{32768x^4} + \dots \right). \quad (23)$$

or, directly, as the square root of corresponding quantiles for the χ_{2pN}^2 distribution.

Figures 3 and 4 show $\eta\%$ confidence intervals for $\text{Avg}(|E_{R,\alpha}|)$ and $\text{Avg}(|E_{R,t}|)$. The features of the plots are qualitatively similar to the corresponding Figs. 1 and 2, although the interval widths now decrease more rapidly with increasing N . Again, the average vector field magnitude yields overall reduced uncertainties, but are decreasing at the same rate compared to a Cartesian component.

Parenthetically, we have considered an ideal chamber with a χ_{2p}^2 distribution for its interior received power. Since a χ_{2pN}^2 distribution approaches Gauss normality for $N \rightarrow +\infty$, the sampling distribution of $\text{Avg}(P_R)$ also asymptotically approaches a Gauss normal form in this limit: for a Gauss normal real X , the sampling pdf of its mean value is, in self-sufficient form,

$$f_{\text{Avg}(X)}[\text{avg}(X); N] = \frac{\exp \left[-\frac{(\text{avg}(X) - \langle \text{Avg}(X) \rangle)^2}{2 \sigma_{\text{Avg}(X)}^2} \right]}{\sqrt{2\pi} \sigma_{\text{Avg}(X)}} \quad (28)$$

with $\langle \text{Avg}(X) \rangle = \langle X \rangle$ and $\sigma_{\text{Avg}(X)} = \sigma_X / \sqrt{N}$. This asymptotic result is a special case of the Lindeberg limit theorem from probability theory, which states that the sample average approaches¹⁹ Gauss normality for $N \rightarrow +\infty$, for general but sufficiently well-behaved pdfs of this sample average [11]. This theorem is a powerful result for use with non-ideal MT/MSRCs, in particular $\chi_q^2(M-1)$ where the ensemble power or field magnitude does not necessarily satisfy a $\chi_{2p}^{(2)}$ pdf.

For reference, Appendix C tabulates calculated percentiles for two-sided 95 % and 99 % confidence intervals of χ_{M-1}^2 distributions with $1 \leq M-1 \leq 10^9$. For $M \rightarrow +\infty$, these distributions approach Gauss normality with mean value $M-1$ and standard deviation $\sqrt{2(M-1)}$, providing an asymptotic 95 % confidence interval given by

$$\left[(M-1) - 1.960 \sqrt{2(M-1)}, \quad (M-1) + 1.960 \sqrt{2(M-1)} \right]. \quad (29)$$

Percentiles for the χ_{M-1} distributed field magnitude follow as $\chi_q(M-1) = \sqrt{\chi_q^2(M-1)}$.

As a final general remark on the statistics of the sample average, the application of the χ_{2pN}^2 distribution to $\text{Avg}(P_R)$ has a sound physical basis in an ideal MT/MSRC, viz., the presence of complete field incoherency. The concepts and results may also be applied, mutatis mutandis, in a partially incoherent (e.g., multipath with line-of-sight) or coherent (anechoic) EME for the estimation of the average field strength, based on N independently generated and *statistically equivalent* field evaluations, as the root of the sum of squares of the in-phase and quadrature field components. In this case, the relevant distribution is the Gauss normal distribution for the complex-valued field E_R with sample statistics listed above.

¹⁹provided the ensemble average and standard deviation exist, i.e., are finite

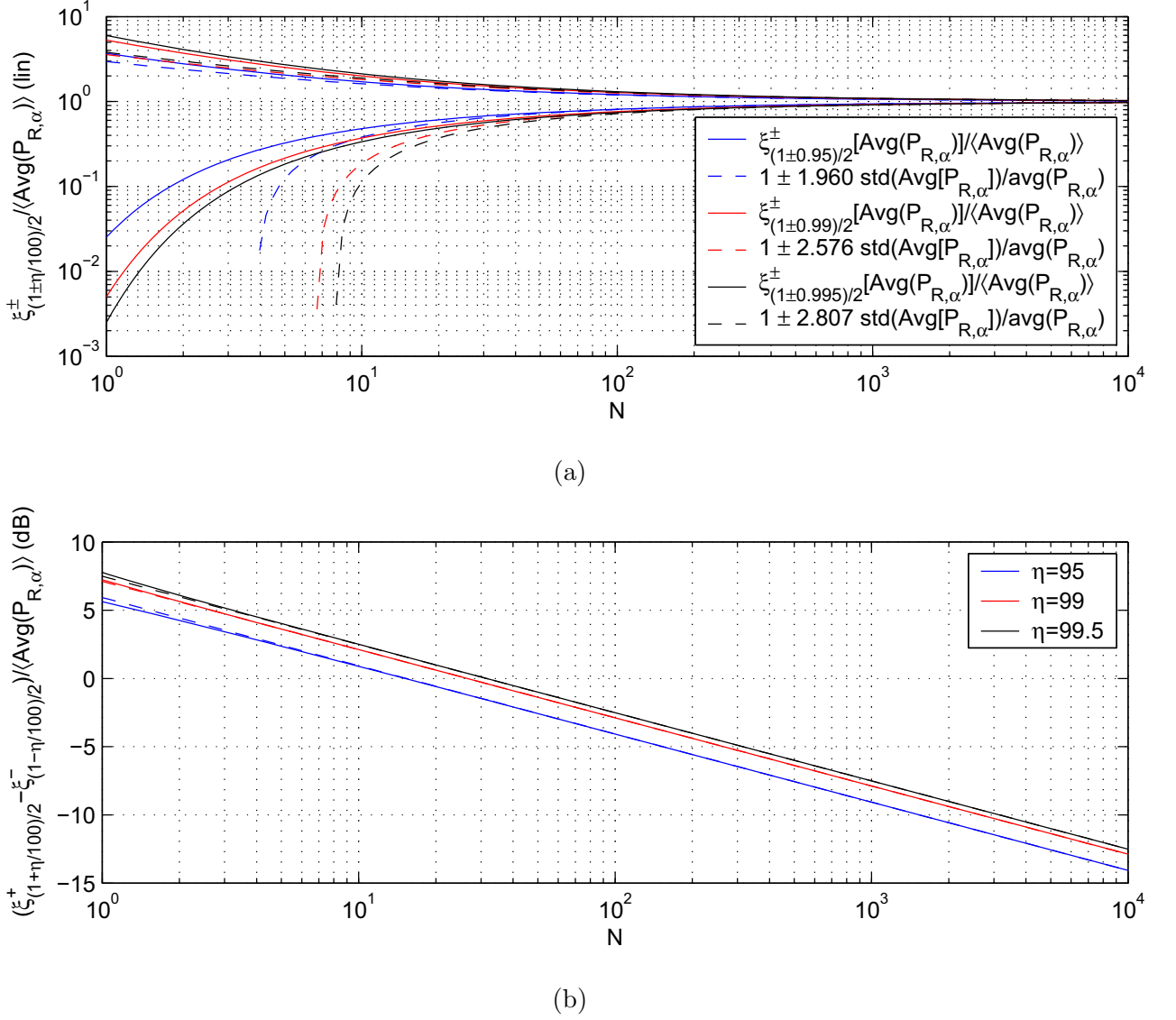


Figure 1: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Avg}(P_{R,\alpha})$ ($p = 1$): (solid lines) exact, based on percentiles of χ_{2N}^2 average-value distribution (for χ_2^2 parent distribution of $P_{R,\alpha}$) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Avg}(P_{R,\alpha})}/\text{avg}(P_{R,\alpha}), 1 + 1.960 s_{\text{Avg}(P_{R,\alpha})}/\text{avg}(P_{R,\alpha})]$, $[1 - 2.576 s_{\text{Avg}(P_{R,\alpha})}/\text{avg}(P_{R,\alpha}), 1 + 2.576 s_{\text{Avg}(P_{R,\alpha})}/\text{avg}(P_{R,\alpha})]$, and $[1 - 2.807 s_{\text{Avg}(P_{R,\alpha})}/\text{avg}(P_{R,\alpha}), 1 + 2.807 s_{\text{Avg}(P_{R,\alpha})}/\text{avg}(P_{R,\alpha})]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB. Dashed blue lines correspond to (144).

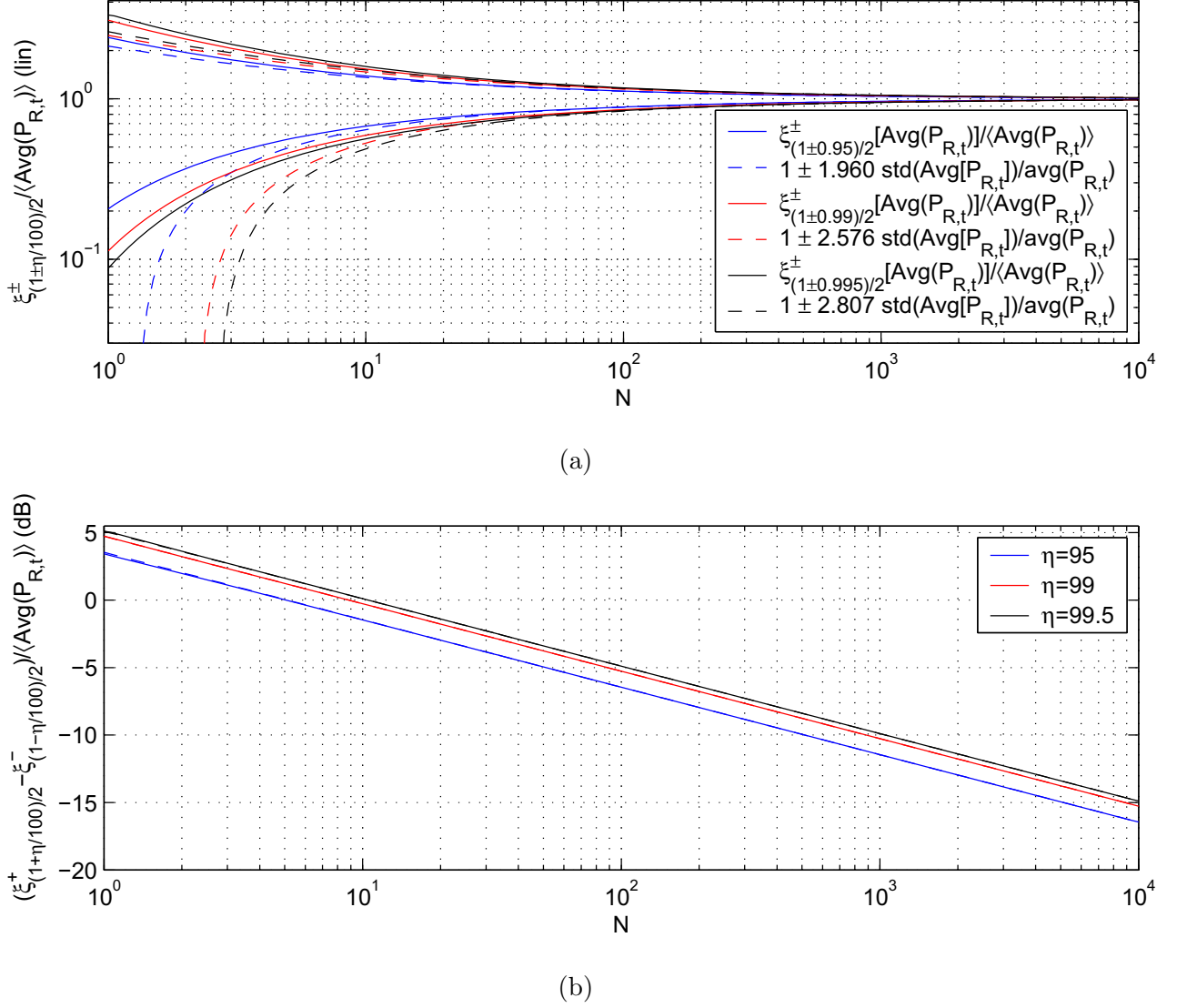


Figure 2: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Avg}(P_{R,t})$ ($p = 3$): (solid lines) exact, based on percentiles of χ_{6N}^2 average-value distribution (for χ_6^2 parent distribution of $P_{R,t}$) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Avg}(P_{R,t})}/\text{avg}(P_{R,t}), 1 + 1.960 s_{\text{Avg}(P_{R,t})}/\text{avg}(P_{R,t})]$, $[1 - 2.576 s_{\text{Avg}(P_{R,t})}/\text{avg}(P_{R,t}), 1 + 2.576 s_{\text{Avg}(P_{R,t})}/\text{avg}(P_{R,t})]$, and $[1 - 2.807 s_{\text{Avg}(P_{R,t})}/\text{avg}(P_{R,t}), 1 + 2.807 s_{\text{Avg}(P_{R,t})}/\text{avg}(P_{R,t})]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB. Dashed blue lines correspond to (145).

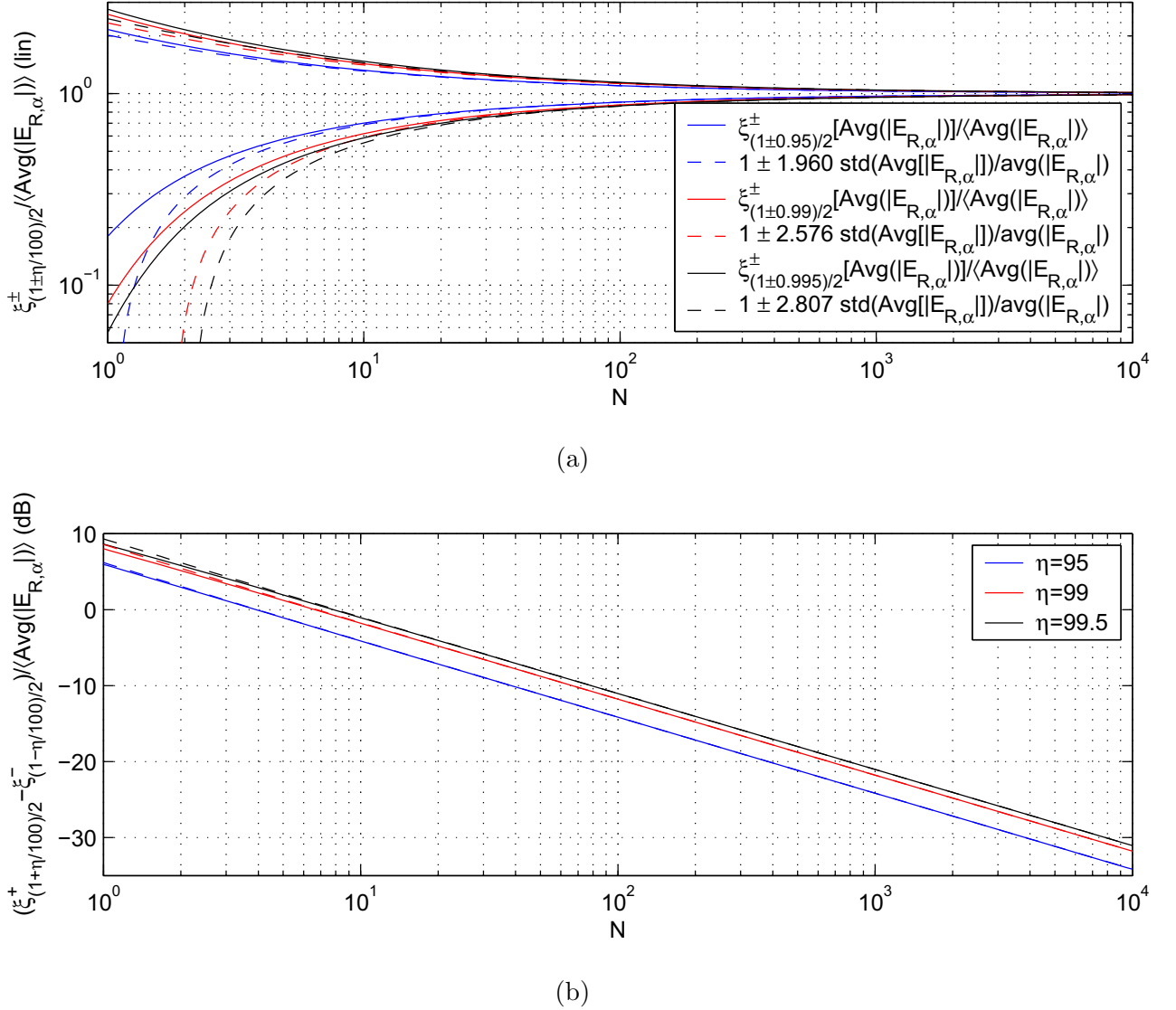


Figure 3: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Avg}(|E_{R,\alpha}|)$ ($p = 1$): (solid lines) exact, based on percentiles of χ_{2N} average-value distribution (for χ_2 parent distribution of $|E_{R,\alpha}|$) with $\eta = 95$, 99, or 99.5; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Avg}(|E_{R,\alpha}|)} / \text{avg}(|E_{R,\alpha}|), 1 + 1.960 s_{\text{Avg}(|E_{R,\alpha}|)} / \text{avg}(|E_{R,\alpha}|)]$, $[1 - 2.576 s_{\text{Avg}(|E_{R,\alpha}|)} / \text{avg}(|E_{R,\alpha}|), 1 + 2.576 s_{\text{Avg}(|E_{R,\alpha}|)} / \text{avg}(|E_{R,\alpha}|)]$, and $[1 - 2.807 s_{\text{Avg}(|E_{R,\alpha}|)} / \text{avg}(|E_{R,\alpha}|), 1 + 2.807 s_{\text{Avg}(|E_{R,\alpha}|)} / \text{avg}(|E_{R,\alpha}|)]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB. Dashed blue lines correspond to (146).

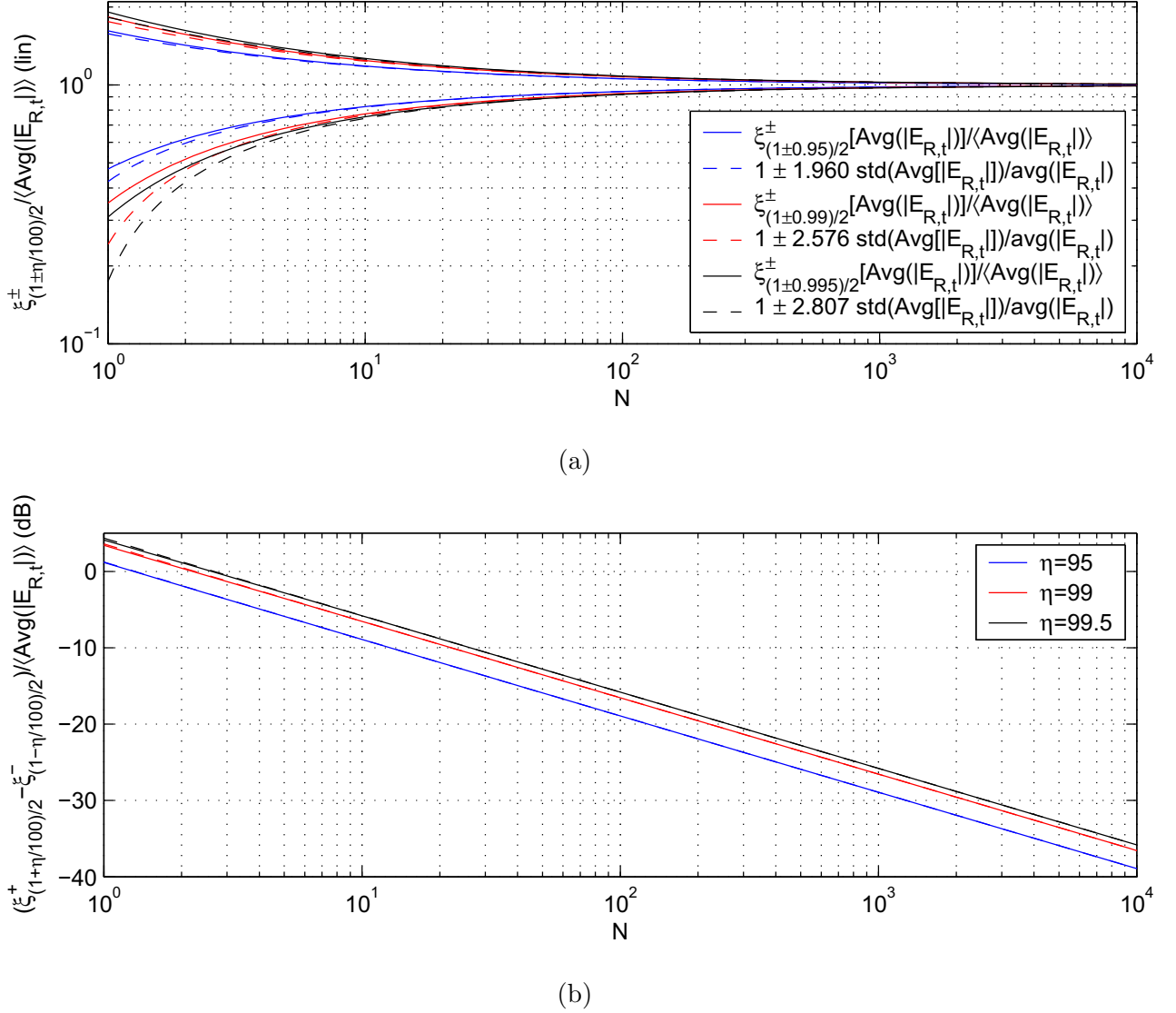


Figure 4: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Avg}(|E_{R,t}|)$ ($p = 3$): (solid lines) exact, based on percentiles of χ_{6N} average-value distribution (for χ_6 parent distribution of $|E_{R,t}|$) with $\eta = 95$, 99, or 99.5; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Avg}(|E_{R,t}|)}/\text{avg}(|E_{R,t}|), 1 + 1.960 s_{\text{Avg}(|E_{R,t}|)}/\text{avg}(|E_{R,t}|)]$, $[1 - 2.576 s_{\text{Avg}(|E_{R,t}|)}/\text{avg}(|E_{R,t}|), 1 + 2.576 s_{\text{Avg}(|E_{R,t}|)}/\text{avg}(|E_{R,t}|)]$, and $[1 - 2.807 s_{\text{Avg}(|E_{R,t}|)}/\text{avg}(|E_{R,t}|), 1 + 2.807 s_{\text{Avg}(|E_{R,t}|)}/\text{avg}(|E_{R,t}|)]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB. Dashed blue lines correspond to (147).

3.1.2 Standard Deviation and Variance

For the estimation of certain field statistics of practical interest (e.g., threshold crossings above or below a critical level, statistics of the number, size, and frequency of occurrence of EM “hot” or “cold” regions in space or time), the ensemble standard deviation σ_X is a quantity of fundamental importance. Its sampling statistic $\text{Std}(X)$, when estimated from a single set of stir sweep data, itself exhibits a standard deviation, the standard error $\sigma_{\text{Std}(X)}$ for all samples of size N (recall that statistical independence of all sample values is hereby required). In other words, $s_{\text{Std}(X)}$ measures the “uncertainty of the uncertainty”²⁰ of samples of X .

Recalling that the m th order moments μ_m and central moments μ'_m of a χ^2_2 distribution for Cartesian $P_{R,\alpha}$ are $\mu_m \triangleq \langle P_{R,\alpha}^m \rangle = m! \sigma_{P_{R,\alpha}}^m$ and $\mu'_m \triangleq \langle (P_{R,\alpha} - \mu_1)^m \rangle$, we obtain after calculation

$$\langle \text{Std}(P_{R,\alpha}) \rangle = \sigma_{P_{R,\alpha}} + \mathcal{O}\left(\frac{\sigma_{P_{R,\alpha}}}{N}\right) \quad (30)$$

$$\sigma_{\text{Std}(P_{R,\alpha})} = \sqrt{\frac{\mu'_4 - \mu'^2_2}{4 \mu'_2 (N-1)}} + \mathcal{O}\left(\frac{\sigma_{P_{R,\alpha}}}{\sqrt{N^3}}\right) = \sqrt{\frac{2}{N-1}} \sigma_{P_{R,\alpha}} + \mathcal{O}\left(\frac{\sigma_{P_{R,\alpha}}}{\sqrt{N^3}}\right) \quad (31)$$

compared to $\langle \text{Std}(X) \rangle \approx \sigma_X$ and $\sigma_{\text{Std}(X)} \simeq \sigma_X / \sqrt{2(N-1)}$ for a Gauss normal real X (cf. (39)), for which $\mu'_2 = \sigma_X^2$, $\mu'_4 = 3\sigma_X^4$. Thus, $\sigma_{\text{Std}(P_{R,\alpha})}$ for a χ^2_2 distributed power $P_{R,\alpha}$ is twice as large as $\sigma_{\text{Std}(X)}$ for its underlying Gauss normal field X . For χ^2_6 distributed vectorial $P_{R,t}$, now with $\mu_m = [(m+2)!/(2 \times 3^{m/2})] \sigma_{P_{R,t}}^m$ and again $\mu'_m \triangleq \langle (P_{R,t} - \mu_1)^m \rangle$, we obtain

$$\sigma_{\text{Std}(P_{R,t})} = \frac{\sigma_{P_{R,t}}}{\sqrt{N-1}} + \mathcal{O}\left(\frac{\sigma_{P_{R,t}}}{\sqrt{N^3}}\right) \quad (32)$$

which is still a factor $\sqrt{2}$ larger than for a corresponding Gauss normal field.

For the sampling variance of $P_{R,\alpha}$, its mean and standard deviation are

$$\langle \text{Var}(P_{R,\alpha}) \rangle = \sigma_{P_{R,\alpha}}^2 + \mathcal{O}\left(\frac{\sigma_{P_{R,\alpha}}^2}{N}\right) \quad (33)$$

$$\sigma_{\text{Var}(P_{R,\alpha})} = \sqrt{\frac{\mu'_4 - \frac{N-3}{N-1} \mu'^2_2}{N-1}} = \sqrt{\frac{\mu'_4 - \mu'^2_2}{N-1}} + \mathcal{O}\left(\frac{\sigma_{P_{R,\alpha}}^2}{\sqrt{N^3}}\right) = \sqrt{\frac{8}{N-1}} \sigma_{P_{R,\alpha}}^2 + \mathcal{O}\left(\frac{\sigma_{P_{R,\alpha}}^2}{\sqrt{N^3}}\right). \quad (34)$$

compared to $\langle \text{Var}(X) \rangle = \sigma_X^2$ and $\sigma_{\text{Var}(X)} = \sqrt{2/(N-1)} \sigma_X^2$ for a Gauss normal real X (cf. (42)). For vectorial $P_{R,t}$,

$$\sigma_{\text{Var}(P_{R,t})} = \frac{2}{\sqrt{N-1}} \sigma_{P_{R,t}}^2 + \mathcal{O}\left(\frac{\sigma_{P_{R,t}}^2}{\sqrt{N^3}}\right). \quad (35)$$

The expressions (31) and (34) are valid for any distribution, at least for the definition (4).

²⁰This somewhat loose statement can be made more precise by calculating confidence intervals for the sampling distribution of the standard deviation, or from the standard error of the confidence interval limits.

For a Gauss normal real X , the sampling pdf of $\text{Std}(X)$ is a scaled χ_{N-1} distribution²¹:

$$f_{\text{Std}(X)}[\text{std}(X); N] = \frac{(N-1)^{[(N-1)/2]}}{2^{[(N-3)/2]} \Gamma\left(\frac{N-1}{2}\right) \sigma_X} \left(\frac{\text{std}(X)}{\sigma_X}\right)^{N-2} \exp\left[-\frac{N-1}{2} \left(\frac{\text{std}(X)}{\sigma_X}\right)^2\right] \quad (36)$$

$$\begin{aligned} &= \frac{2}{\Gamma\left(\frac{N-1}{2}\right) \sigma_{\text{Std}(X)}} \left[\frac{N-1}{2} - \left(\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} \right)^2 \right]^{\frac{N-1}{2}} \\ &\quad \times \left(\frac{\text{std}(X)}{\sigma_{\text{Std}(X)}} \right)^{N-2} \exp\left\{ - \left[\frac{N-1}{2} - \left(\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} \right)^2 \right] \left(\frac{\text{std}(X)}{\sigma_{\text{Std}(X)}} \right)^2 \right\} \end{aligned} \quad (37)$$

for $N \geq 2$, with m th order moments

$$\langle [\text{Std}(X)]^m \rangle = \sqrt{\left(\frac{2}{N-1}\right)^m} \frac{\Gamma\left(\frac{N-1+m}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} \sigma_X^m. \quad (38)$$

In particular, the sampling mean and standard deviation of $\text{Std}(X)$ and their asymptotic approximations for $N \rightarrow +\infty$ follow with the aid of (23) as

$$\langle \text{Std}(X) \rangle = \sigma_X \sqrt{\frac{2}{N-1}} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} \rightarrow \sqrt{\frac{2N-3}{2N-2}} \sigma_X = \sigma_X + \mathcal{O}\left(\frac{\sigma_X}{\sqrt{N-1}}\right), \quad (39)$$

$$\sigma_{\text{Std}(X)} = \sigma_X \sqrt{1 - \frac{2}{N-1} \left(\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} \right)^2} \rightarrow \frac{\sigma_X}{\sqrt{2(N-2)}} = \frac{\sigma_X}{\sqrt{2(N-1)}} + \mathcal{O}\left(\frac{\sigma_X}{N-1}\right). \quad (40)$$

Note that $\langle \text{Std}(X) \rangle < \sigma_X$ for any $N > 1$, although the difference between $\langle \text{Std}(X) \rangle$ and σ_X vanishes for $N \rightarrow +\infty$. In other words, the factor in (39) yields an unbiased estimate of σ_X .

Upon the variate transformation $\text{Var}(X) = [\text{Std}(X)]^2$, the sampling variance of X follows from (36) as a scaled χ_{N-1}^2 variate:

$$f_{\text{Var}(X)}[\text{var}(X); N] = \frac{(N-1)^{[(N-1)/2]}}{2^{[(N-1)/2]} \Gamma\left(\frac{N-1}{2}\right) \sigma_X^2} \left(\frac{\text{var}(X)}{\sigma_X^2} \right)^{\frac{N-3}{2}} \exp\left(-\frac{N-1}{2} \frac{\text{var}(X)}{\sigma_X^2}\right) \quad (41)$$

whose mean and variance are

$$\langle \text{Var}(X) \rangle = \sigma_X^2, \quad \sigma_{\text{Var}(X)}^2 = \frac{2}{N-1} \sigma_X^4. \quad (42)$$

Calculation of the pdfs of the standard error and sampling variance of the $\chi^{(2)}$ distributed P_R or $|E_R|$ is more complicated than for a Gauss normal real X and follows using variate transformations

²¹NB: In (37)–(42), an unbiased estimate of s_X^2 for σ_X^2 (corresponding to a reduced bias of s_X for σ_X) is assumed; cf. (6). If, for example, $N-1$ in the denominator of (4) were replaced by N , then σ_X in (37)–(42) should be replaced accordingly by $\sigma_X \sqrt{(N-1)/N}$. As noted before, the estimate $s_{\text{Std}(X)}$ becomes nearly unbiased if $N-1$ in (4) is replaced with $N-\frac{3}{2}$, in which case (39) and (40) become $\langle \text{Std}(X) \rangle = \sigma_X \sqrt{2/(N-\frac{3}{2})} \Gamma(\frac{N}{2})/\Gamma(\frac{N-1}{2})$ and $\sigma_{\text{Std}(X)} = \sigma_X \sqrt{(N-1)/(N-\frac{3}{2}) - [2/(N-\frac{3}{2})][\Gamma(\frac{N}{2})/\Gamma(\frac{N-1}{2})]^2}$ that yield marginally higher values than (39) and (40).

on circular normal E_R , with the aid of (3) and (15) or (25). On account of (10) and (41), $\text{Std}(P_R)$ exhibits a χ^2_{2pN-1} pdf:

$$\begin{aligned} f_{\text{Std}(P_R)}[\text{std}(P_R); N] &= \frac{\left(pN - \frac{1}{2}\right)^{pN - \frac{1}{2}}}{\Gamma\left(pN - \frac{1}{2}\right) \sigma_{P_R}} \left(\frac{\text{std}(P_R)}{\sigma_{P_R}}\right)^{pN - \frac{3}{2}} \exp\left[-\left(pN - \frac{1}{2}\right) \frac{\text{std}(P_R)}{\sigma_{P_R}}\right] \\ &= \frac{\left(pN - \frac{1}{2}\right)^{\frac{1}{2}(pN - \frac{1}{2})}}{\Gamma\left(pN - \frac{1}{2}\right) \sigma_{\text{Std}(P_R)}} \left(\frac{\text{std}(P_R)}{\sigma_{\text{Std}(P_R)}}\right)^{pN - \frac{3}{2}} \exp\left(-\sqrt{pN - \frac{1}{2}} \frac{\text{std}(P_R)}{\sigma_{\text{Std}(P_R)}}\right). \end{aligned} \quad (43)$$

Similarly, from (24) and (37), $\text{Std}(|E_R|)$ exhibits a χ^2_{2pN-1} pdf:

$$\begin{aligned} f_{\text{Std}(|E_R|)}[\text{std}(|E_R|); N] &= \frac{2 \left[pN - \frac{1}{2} - \left(\frac{\Gamma(pN)}{\Gamma(pN - \frac{1}{2})}\right)^2\right]^{pN - \frac{1}{2}}}{\Gamma(pN - \frac{1}{2}) \sigma_{\text{Std}(|E_R|)}} \left(\frac{\text{std}(|E_R|)}{\sigma_{\text{Std}(|E_R|)}}\right)^{2(pN-1)} \\ &\quad \times \exp\left\{-\left[pN - \frac{1}{2} - \left(\frac{\Gamma(pN)}{\Gamma(pN - \frac{1}{2})}\right)^2\right] \left(\frac{\text{std}(|E_R|)}{\sigma_{\text{Std}(|E_R|)}}\right)^2\right\}. \end{aligned} \quad (44)$$

Recall that, on account of the central limit theorem, these distributions also tend to Gauss normality when $N \rightarrow +\infty$, and even more rapidly so than those for a Gauss normal real X and for the sampling distribution of the mean value, because of the now larger number of degrees of freedom. Confidence intervals for $\text{Std}(P_R)$ and $\text{Std}(|E_R|)$ follow from Tbl. 1 and Fig. 5 upon substituting $N \rightarrow 2pN$. Also, with this substitution, sampling statistics follow as

$$\langle \text{Std}(P_R) \rangle = \sigma_{P_R}, \quad \sigma^2_{\text{Std}(P_R)} = \frac{\sigma_{P_R}^2}{pN - \frac{1}{2}} \quad (45)$$

and

$$\langle \text{Std}(|E_R|) \rangle = \frac{\sigma_{|E_R|}}{\sqrt{pN - \frac{1}{2}}} \frac{\Gamma(pN)}{\Gamma(pN - \frac{1}{2})} \rightarrow \sigma_{|E_R|} + \mathcal{O}\left(\frac{\sigma_{|E_R|}}{\sqrt{pN - \frac{1}{2}}}\right), \quad (46)$$

$$\sigma_{\text{Std}(|E_R|)} = \sigma_{|E_R|} \sqrt{1 - \frac{1}{pN - \frac{1}{2}} \left(\frac{\Gamma(pN)}{\Gamma(pN - \frac{1}{2})}\right)^2} \rightarrow \frac{\sigma_{|E_R|}}{2\sqrt{pN - \frac{1}{2}}} + \mathcal{O}\left(\frac{\sigma_{|E_R|}}{pN - \frac{1}{2}}\right). \quad (47)$$

The ensemble standard deviation of a circular Gauss normal distributed complex field is a fundamental quantity, from which distributions and statistics of power, intensity and magnitude all follow. Hence a detailed numerical investigation of the uncertainty for its estimation is of basic interest. To this end, Tbl. 1 lists values of $\sigma_{\text{Std}(X)}$ for a Gauss normal real X , standardized (i.e., divided) by σ_X , together with exact and approximate boundaries and width of a 95 % confidence interval of $\text{Std}(X)$. These boundaries and width are presented in Fig. 5. Even for moderately large N , the relative width of the 95 % confidence interval of $\text{Std}(X)$ is seen to be considerable: for example, for $N = 30, 100$, and 300 , this width is 51.2 %, 27.8 %, and 16.0 % of the central value σ_X of this interval, respectively.

(Further calculations show that one requires about $N > 800$ or 80 000 in order to reduce the interval width to less than 10 % or 1 %, respectively.) For $N \rightarrow +\infty$, it is numerically verified from the Table that the width of the confidence interval approaches asymptotically $2 \times 1.960/\sqrt{2(N-1)}$, as expected from (40) and application of the central limit theorem.

Note that for a χ_2^2 or χ_6^2 distributed P_R , the values of $\sigma_{\text{Std}(P_R)}/\sigma_{P_R}$ are larger than those of $\sigma_{\text{Std}(X)}/\sigma_X$. For large N , the associated interval widths follow approximately by multiplying the values in the last column of Tbl. 1 by a factor 2 or $\sqrt{2}$, respectively, as noted before.

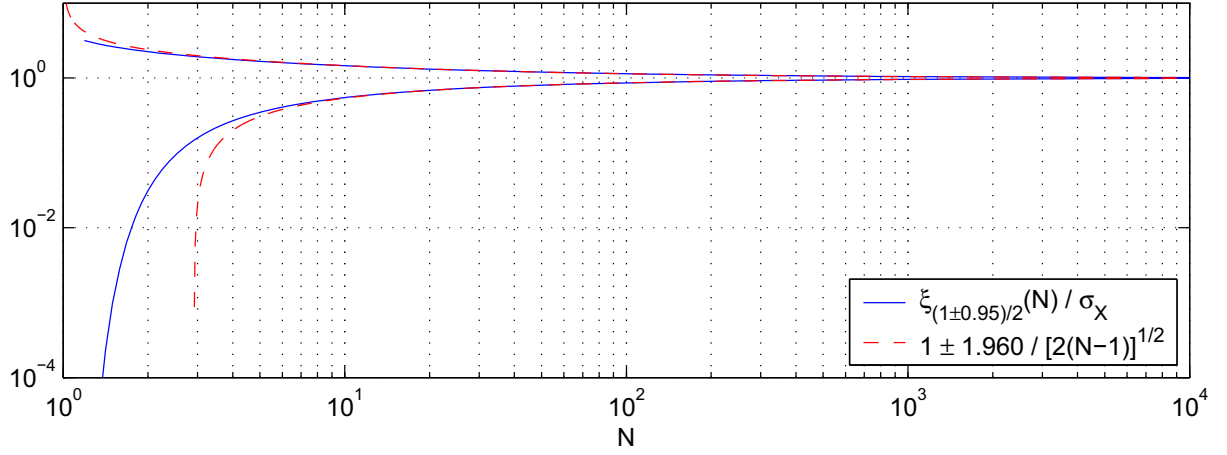
Table 1: Limits and width of normalized two-sided 95 % confidence interval of $\text{Std}(X)$, for samples of size N from a Gauss normal distributed X . Exact width $[\chi_{0.975}(N) - \chi_{0.025}(N)]/\sigma_X$ is based on 95% percentiles for symmetric confidence interval (4th column) and compared with two approximations based on either the expanded normalized standard error, i.e., $2 \times 1.960 \sigma_{\text{Std}(X)}/\langle \text{Std}(X) \rangle$ (5th column) or the asymptotic value of $2 \times 1.960 \sigma_{\text{Std}(X)}/\sigma_X$ (6th column). For samples of size N for $\text{Std}(|E_R|)$, corresponding interval limits $\chi_q(2pN)/\sigma_{|E_R|}$ and widths follow by replacing N by $2pN$ for $q = 0.025, 0.975$; for $\text{Std}(P_R)$, corresponding limits $\chi_q^2(2pN)/\sigma_{P_R}$ and widths follow by replacing N by $2pN$ and squaring the normalized percentiles.

N	$\frac{\chi_{0.025}(N)}{\sigma_X}$	$\frac{\chi_{0.975}(N)}{\sigma_X}$	$\frac{\chi_{0.975}(N) - \chi_{0.025}(N)}{\sigma_X}$	$2 \times 1.960 \sigma_{\text{Std}(X)}/\langle \text{Std}(X) \rangle$ $= 2 \times 1.960 \frac{\sqrt{1 - \frac{2}{N-1} \left(\frac{\Gamma(N/2)}{\Gamma[(N-1)/2]} \right)^2}}{\sqrt{\frac{2}{N-1} \frac{\Gamma(N/2)}{\Gamma[(N-1)/2]}}}$ $= 2 \times 1.960 \frac{\sigma_{\text{Std}(X)}}{\sigma_X} \sqrt{\frac{N-1}{2} \frac{\Gamma[(N-1)/2]}{\Gamma(N/2)}}$	$2 \times 1.960/\sqrt{2(N-1)}$ $\simeq 2 \times 1.960 \sigma_{\text{Std}(X)}/\sigma_X$ $\simeq 2 \times 1.960 \frac{\sigma_{\text{Std}(X)}}{\langle \text{Std}(X) \rangle}$
2	0.0313	2.241	2.210	2.962	2.772
3	0.159	1.921	1.762	2.049	1.960
4	0.268	1.765	1.497	1.654	1.600
5	0.348	1.669	1.321	1.423	1.386
6	0.408	1.602	1.194	1.267	1.240
7	0.454	1.552	1.098	1.153	1.132
8	0.491	1.512	1.021	1.065	1.048
9	0.522	1.481	0.959	0.994	0.980
10	0.548	1.454	0.906	0.936	0.924
12	0.589	1.412	0.823	0.845	0.836
16	0.646	1.354	0.708	0.721	0.716
20	0.685	1.315	0.630	0.640	0.635
24	0.713	1.287	0.574	0.581	0.578
30	0.744	1.256	0.512	0.517	0.515
36	0.767	1.233	0.466	0.470	0.469
40	0.779	1.221	0.442	0.445	0.444
48	0.798	1.201	0.403	0.405	0.404
50	0.802	1.197	0.395	0.397	0.396
60	0.820	1.180	0.360	0.362	0.361
72	0.836	1.164	0.328	0.330	0.329
80	0.844	1.156	0.311	0.312	0.312
96	0.858	1.142	0.284	0.285	0.284
100	0.8608	1.1389	0.2781	0.2786	0.2786
120	0.8730	1.1268	0.2538	0.2544	0.2541
200	0.9018	1.0981	0.1963	0.1967	0.1965
300	0.9199	1.0801	0.1602	0.1604	0.1603
500	0.9380	1.0620	0.1240	0.1242	0.1241
600	0.9438	1.0566	0.1129	0.1134	0.1133
720	0.9483	1.0517	0.1034	0.1035	0.1034
1000	0.9562	1.0438	0.0876	0.0877	0.0877
10^4	0.98614	1.01386	0.02772	0.02772	0.02772
10^5	0.99562	1.00438	0.00877	0.00877	0.00877
10^6	0.998614	1.001386	0.002772	0.002772	0.002772

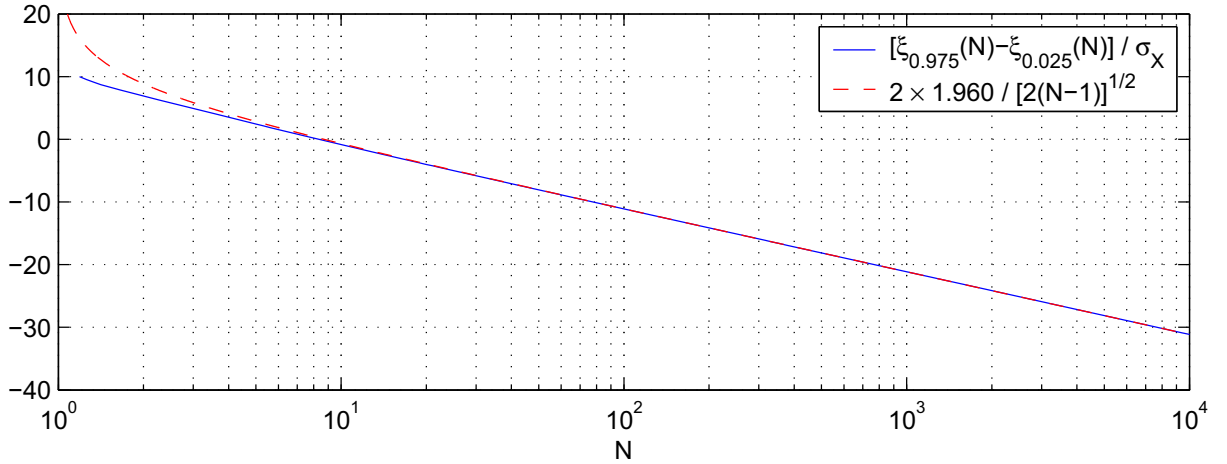
It can be concluded that, for all realistic values of N , the relative uncertainty of $\text{Std}(X)$ is substantial and can be a major contribution to the IFU in practice.

For later use, we note that the sample average and sample standard deviation are statistically

independent only in the case of a Gauss normal distribution. For general distributions, their covariance is, to order $1/N$, proportional to the coefficient of skewness. Hence, in particular, for any symmetric pdf, the sample average and the sample standard deviation are uncorrelated (i.e., linearly independent) [11, Sec. 29.3].



(a)



(b)

Figure 5: (a) Mean-normalized boundaries of 95% confidence intervals of $\text{Std}(X)$ for Gauss normal real X : (solid blue lines) exact, based on percentiles of χ_{N-1} distribution of $\text{Std}(X)$; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 \sigma_{\text{Std}(X)} / \sigma_X, 1 + 1.960 \sigma_{\text{Std}(X)} / \sigma_X]$. (b) Associated width (mean-normalized difference) of 95% confidence interval for $X = |E|$, in dB. [For $X = |E|^2$, the dB values must be divided by a factor two.]

3.1.3 Coefficient of Variation

The coefficient of variation $\nu_{P_R} \triangleq \sigma_{P_R}/\mu_{P_R}$ (also known as the normalized or relative standard deviation), is a measure for the “contrast” that exists between “hot” and “cold” locations in the random temporal or spatial pattern of the EM power or intensity (or, similarly, for the field magnitude). Its inverse represents the inverse of a “signal-to-noise” ratio for field nonhomogeneity “hidden” within a sea of random fluctuations.

For the sampling mean and standard deviation of the sample values n_{P_R} of ν_{P_R} , as an estimate for the ensemble coefficient ν_{P_R} , using the expressions for $\mu_m^{(j)}$ of Sec. 3.1.2, we obtain

$$\langle n_{P_R,\alpha} \rangle = \nu_{P_R,\alpha} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \quad (48)$$

$$\sigma_{n_{P_R,\alpha}} = \frac{\nu_{P_R,\alpha}}{\sqrt{N-1}} \sqrt{\frac{\mu'_4 - \mu'^2_2}{4 \mu'^2_2} + \frac{\mu'_2}{\mu'^2_1} - \frac{\mu'_3}{\mu'_2 \mu_1}} \quad (49)$$

$$= \frac{1}{\sqrt{N-1}} + \mathcal{O}\left(\frac{1}{N}\right). \quad (50)$$

The latter is larger than the corresponding value $(\nu_X/\sqrt{2(N-1)})\sqrt{1+2\nu_X^2}$ for a (noncentral) Gauss normal distributed X by a factor $\sqrt{3/2} \simeq 1.225$. For vectorial $P_{R,t}$,

$$\sigma_{n_{P_R,t}} = \frac{2}{3\sqrt{3(N-1)}}. \quad (51)$$

The ratio $\sigma_{n_{P_R,t}}/\sigma_{n_{P_R,\alpha}} = 2/(3\sqrt{3})$ is 33 % smaller than the corresponding ratio of averages or expected values, viz., $\langle n_{P_R,t} \rangle / \langle n_{P_R,\alpha} \rangle = \nu_{P_R,t} / \nu_{P_R,\alpha} = 1/\sqrt{3}$.

3.1.4 Boundaries of 95 % and 99 % Confidence Intervals

Just like any other sample statistic, the upper (ξ^+) and lower (ξ^-) boundaries of an η % confidence interval with equiprobabilities, i.e., $\xi_{(1\pm\eta/100)/2}^\pm$ for the ensemble of P_R values acquire uncertainty due to fluctuations of the sampling distribution of P_R : only the boundaries of the ensemble pdf have deterministic (fixed) locations. The boundaries for sampling variates and sampling distributions are random variables and here are represented²² as $\Xi_{(1\pm\eta/100)/2}^\pm$.

For a 95 % confidence interval of the χ^2_2 distributed $P_{R,\alpha}$, the standard errors and covariance²³ of $\Xi_{(1\pm\eta/100)/2,\alpha}^\pm$ are

$$\sigma_{\Xi_{0.975,\alpha}^+} = \sqrt{\frac{0.025 \times 0.975}{[f_{P_R,\alpha}(F_{P_R,\alpha}^{-1}(0.975))]^2 N}} = \sqrt{\frac{39}{N}} \sigma_{P_R,\alpha} \simeq \frac{6.245}{\sqrt{N}} \sigma_{P_R,\alpha} = \frac{1.693}{\sqrt{N}} \xi_{0.975,\alpha}^+ \quad (52)$$

²²We shall exceptionally use the same symbol ξ to denote values of both ensemble- and sample-based quantiles. It will be clear from the context whether the former or the latter is meant. Sample and ensemble quantiles agree to order $1/N$.

²³Note that, in general, σ_{Ξ^+,Ξ^-} in (54) is asymmetric with respect to upper and lower percentile values.

$$\sigma_{\Xi_{0.025,\alpha}^-} = \sqrt{\frac{0.025 \times 0.975}{[f_{P_{R,\alpha}}(F_{P_{R,\alpha}}^{-1}(0.025))]^2 N}} = \frac{1}{\sqrt{39 N}} \sigma_{P_{R,\alpha}} \simeq \frac{0.160}{\sqrt{N}} \sigma_{P_{R,\alpha}} = \frac{6.324}{\sqrt{N}} \xi_{0.025,\alpha}^- \quad (53)$$

$$\sigma_{\Xi_{0.975,\alpha}^+, \Xi_{0.025,\alpha}^-} = \frac{(0.975)^2}{f_{P_{R,\alpha}}(F_{P_{R,\alpha}}^{-1}(0.025)) f_{P_{R,\alpha}}(F_{P_{R,\alpha}}^{-1}(0.975)) N} = \frac{39}{N} \sigma_{P_{R,\alpha}}^2 = \frac{417.876}{N} \xi_{0.975,\alpha}^+ \xi_{0.025,\alpha}^- \quad (54)$$

with $\xi_{0.975,\alpha}^+ = -[\ln(1 - 0.975)] \sigma_{P_{R,\alpha}} \simeq 3.6889 \sigma_{P_{R,\alpha}}$, $\xi_{0.025,\alpha}^- = -[\ln(1 - 0.025)] \sigma_{P_{R,\alpha}} \simeq 0.0253 \sigma_{P_{R,\alpha}}$. From (52)–(54), one may formally define²⁴ a cross-correlation coefficient between the upper and lower boundaries as

$$\rho_{\Xi_{0.975,\alpha}^+, \Xi_{0.025,\alpha}^-} \triangleq \frac{\sigma_{\Xi_{0.975,\alpha}^+, \Xi_{0.025,\alpha}^-}}{\sigma_{\Xi_{0.975,\alpha}^+} \sigma_{\Xi_{0.025,\alpha}^-}} = \frac{0.975}{0.025} = 39. \quad (55)$$

Its value is independent of the functional form of the parent distribution, but is governed by the location and width of the confidence interval.

For a 99 % confidence interval for $P_{R,\alpha}$, the standard deviations are $\sigma_{\Xi_{0.995,\alpha}^+} = (\sqrt{199/N}) \sigma_{P_{R,\alpha}} \simeq (14.107/\sqrt{N}) \sigma_{P_{R,\alpha}} = (2.663/\sqrt{N}) \xi_{0.995,\alpha}^+$ and $\sigma_{\Xi_{0.005,\alpha}^-} = (1/\sqrt{199 N}) \sigma_{P_{R,\alpha}} \simeq (0.071/\sqrt{N}) \sigma_{P_{R,\alpha}} = (14.200/\sqrt{N}) \xi_{0.005,\alpha}^-$; the covariance is $\sigma_{\Xi_{0.995,\alpha}^+, \Xi_{0.005,\alpha}^-} = (199/N) \sigma_{P_{R,\alpha}}^2 = (7511.8/N) \xi_{0.995,\alpha}^+ \xi_{0.005,\alpha}^-$, with $\xi_{0.995,\alpha}^+ \simeq 5.298 \sigma_{P_{R,\alpha}}$ and $\xi_{0.005,\alpha}^- \simeq 0.0050 \sigma_{P_{R,\alpha}}$. The cross-correlation is $\rho_{\Xi_{0.995,\alpha}^+, \Xi_{0.005,\alpha}^-} = 199$.

Comparing the interval boundaries for the 95 % vs. 99 % confidence levels, and noting that the 95 % interval is entirely comprised within the 99 % interval, it follows that $\sigma_{\Xi_{\alpha}^{\pm}}$ increases in absolute terms with the ensemble value of $\xi_{\alpha}^{\pm}$, i.e., with increasing right-shift of the boundaries of the interval.

For the χ_6^2 distributed vectorial $P_{R,t}$, the sampling standard deviations for the boundaries of a 95 % interval are

$$\sigma_{\Xi_{0.975,t}^+} = \sqrt{\frac{0.025 \times 0.975}{[f_{P_{R,t}}(F_{P_{R,t}}^{-1}(0.975))]^2 N}} \simeq \frac{4.738}{\sqrt{N}} \sigma_{P_{R,t}} = \frac{1.136}{\sqrt{N}} \xi_{0.975,t}^+ \quad (56)$$

$$\sigma_{\Xi_{0.025,t}^-} = \sqrt{\frac{0.025 \times 0.975}{[f_{P_{R,t}}(F_{P_{R,t}}^{-1}(0.025))]^2 N}} \simeq \frac{0.875}{\sqrt{N}} \sigma_{P_{R,t}} = \frac{2.450}{\sqrt{N}} \xi_{0.025,t}^- \quad (57)$$

$$\sigma_{\Xi_{0.975,t}^+, \Xi_{0.025,t}^-} = \frac{(0.975)^2}{f_{P_{R,t}}(F_{P_{R,t}}^{-1}(0.025)) f_{P_{R,t}}(F_{P_{R,t}}^{-1}(0.975)) N} \simeq \frac{161.726}{N} \sigma_{P_{R,t}}^2 = \frac{108.577}{N} \xi_{0.975,t}^+ \xi_{0.025,t}^- \quad (58)$$

with $\xi_{0.975,t}^+ \simeq 4.1711 \sigma_{P_{R,t}}$, $\xi_{0.025,t}^- \simeq 0.3571 \sigma_{P_{R,t}}$ and again

$$\rho_{\Xi_{0.975,t}^+, \Xi_{0.025,t}^-} = 39. \quad (59)$$

²⁴Since the thus defined coefficient yields values between 0 and $+\infty$, it does not produce normalized values (as a typical definition of cross-correlation coefficient would instead), for reasons explained in the previous footnote. However, it does provide a dimensionless quantity.

For a 99 % interval for $P_{R,t}$, the corresponding expressions are $\sigma_{\Xi_{0.995,t}^+} \simeq (10.092/\sqrt{N})\sigma_{P_{R,t}}$, $\sigma_{\Xi_{0.005,t}^-} \simeq (0.9996/\sqrt{N})\sigma_{P_{R,t}} = (5.123/\sqrt{N})\xi_{0.005,t}^-$, and $\sigma_{\Xi_{0.995,t}^+, \Xi_{0.005,t}^-} \simeq (2007.58/N)\sigma_{P_{R,t}}^2 = (1921.8/N)\xi_{0.995,t}^+\xi_{0.005,t}^-$ with again $\rho_{\Xi_{0.995,t}^+, \Xi_{0.005,t}^-} = 199$.

Comparing (52)–(54) with (56)–(58) and noting that $\sigma_{P_{R,t}}^2 = 3\sigma_{P_{R,\alpha}}^2$, it follows that for a specified confidence level, the boundaries of the confidence interval for $P_{R,t}$ exhibit a smaller uncertainty than those for $P_{R,\alpha}$.

One may also define sample confidence intervals with associated uncertainties for sample statistics themselves, e.g., for $\text{Avg}(P_{R,\alpha})$, by noting that for sufficiently large N , the values of $\text{Avg}(P_{R,\alpha})$ exhibit a Gauss normal distribution. Such intervals will be given in Sec. 4.2 for the average and standard deviation of Cartesian and vector power and field.

Finally, for completeness, we calculate and list the sampling standard deviation for the corresponding percentiles of a centered Gauss normally distributed X . For the 95 % confidence interval, we have

$$\sigma_{\Xi_{0.975}^+} = \sigma_{\Xi_{0.025}^-} = \sqrt{\frac{0.025 \times 0.975}{[f_X(F_X^{-1}(0.975))]^2 N}} \simeq \frac{2.671}{\sqrt{N}}\sigma_X = \frac{1.363}{\sqrt{N}}\xi_{0.975}^+ = -\frac{1.363}{\sqrt{N}}\xi_{0.025}^- \quad (60)$$

$$\sigma_{\Xi_{0.975}^+, \Xi_{0.025}^-} = \frac{(0.975)^2}{f_X(F_X^{-1}(0.025)) f_X(F_X^{-1}(0.975)) N} \simeq \frac{278.298}{N}\sigma_X^2 = -\frac{72.446}{N}\xi_{0.975}^+\xi_{0.025}^- \quad (61)$$

with $f_X(\xi_{0.975}^+ = -\xi_{0.025}^- = 1.960\sigma_X) \simeq 0.05845$. For the 99 % interval, we obtain $\sigma_{\Xi_{0.995}^+} = \sigma_{\Xi_{0.005,t}^-} \simeq (4.878/\sqrt{N})\sigma_X = (1.894/\sqrt{N})\xi_{0.995}^+ \simeq -(1.894/\sqrt{N})\xi_{0.005}^-$, and $\sigma_{\Xi_{0.995}^+, \Xi_{0.005}^-} \simeq (4735.05/N)\sigma_X^2 = -(713.663/N)\xi_{0.995}^+\xi_{0.005}^-$, with $f_X(\xi_{0.995}^+ = -\xi_{0.005}^- = 2.576\sigma_X) \simeq 0.01446$.

3.1.5 Extreme Values

The maximum and minimum values of a sample set of size N are sample statistics of particular interest in emission and immunity testing, e.g., as the values of field intensity or power at EM “hot” or “cold” regions in space or time. Minimum values are relevant to communication drop-outs caused by fading or more general multipath interference in propagation. The distributions of these sample statistics are known exactly but are not based on moments, i.e., they do not rely on invoking the central limit theorem for large N (even though this approximation remains of course valid in such cases).

For a χ_2^2 parent distribution, the sampling standard deviations for $\text{Max}(P_{R,\alpha})$ and $\text{Min}(P_{R,\alpha})$ are [15, Sec. 3.2.6], [16, App. A]

$$\sigma_{\text{Max}(P_{R,\alpha})} \simeq \sigma_{P_{R,\alpha}} \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}} \quad (62)$$

$$\sigma_{\text{Min}(P_{R,\alpha})} = \frac{\sigma_{P_{R,\alpha}}}{N}. \quad (63)$$

Upper and lower percentiles for 95 % confidence intervals of the maximum and minimum values are

$$\xi_{0.975}^+(\text{Max}(P_{R,\alpha})) = \ln[(1 - 0.975)^{1/N}]^{-1} \sigma_{P_{R,\alpha}} \quad (64)$$

$$\xi_{0.025}^-(\text{Max}(P_{R,\alpha})) = \ln[(1 - 0.025)^{1/N}]^{-1} \sigma_{P_{R,\alpha}} \quad (65)$$

$$\xi_{0.975}^+(\text{Min}(P_{R,\alpha})) = \frac{3.6889}{N} \sigma_{P_{R,\alpha}} \quad (66)$$

$$\xi_{0.025}^-(\text{Min}(P_{R,\alpha})) = \frac{0.0253}{N} \sigma_{P_{R,\alpha}}. \quad (67)$$

For $\text{Max}(P_{R,t})$ and $\text{Min}(P_{R,t})$ associated with a χ_6^2 parent distribution, the sampling standard deviation and percentiles are more difficult to express analytically. We defer further numerical characterization of the statistics of the maximum value of $P_{R,t}$ to Sec. 5.3.6.1.

3.2 Realistic Chamber ($N_{\max} \not\rightarrow +\infty$)

In practice, for finite-sized chambers and finite frequencies of operation, i.e., for finite mode densities and mode counts, the maximum possible number of independent stir states that can be physically generated, $N_{\max}$, is finite because the correlation length (whether considered in the spatial or in the stir domain) is nonvanishing. Of particular importance is the case where $N \not\ll N_{\max}$. For example, at some arbitrary frequency, a mode tuner may be capable of generating up to, say, $N_{\max} = 30$ statistically independent sample values of the received power, whilst the EMC test or experiment is designed to generate just $N = 12$ actual stir states among these, for reason of economy. Whether or not an additional stirring mechanism can be applied that would increase the size of the sample space and, hence, $N_{\max}$ is not pertinent here: the sample set is taken from the particular (1-D) generated random process which has some definite value of $N_{\max}$ associated with it. The practical determination of the value of $N_{\max}$ is beyond the scope of this report.

For finite $N_{\max}$, the sample mean value is unaffected, but the mean of the sampling variance is now [12]

$$\langle \text{Var}(P_R) \rangle = \frac{N_{\max}}{N_{\max} - 1} \sigma_{P_R}^2 \quad (68)$$

compared to (33). For the sample standard deviation,

$$\langle \text{Std}(P_R) \rangle \simeq \sqrt{\frac{N_{\max}}{N_{\max} - 1}} \sqrt{\frac{N - \frac{3}{2}}{N}} \sigma_{P_R}. \quad (69)$$

The sampling variance of $\text{Avg}(P_R)$ now becomes after calculation [12], instead of (3),

$$\sigma_{\text{Avg}(P_R)}^2 = \frac{N_{\max} - N}{N_{\max} - 1} \frac{\sigma_{P_R}^2}{N}. \quad (70)$$

Defining an equivalent number of samples N' via $s_{\text{Avg}(P_R)}^2 \triangleq \sigma_{P_R}^2/N'$, valid for a sample set taken from a *finite*-sized ensemble, it follows that confidence intervals of $\text{Avg}(P_R)$ for infinite ensembles can be formally generalized to finite-sized ensembles, to order $1/N$, by replacing N by the larger value

$$N' \triangleq \frac{N_{\max} - 1}{N_{\max} - N} N \quad (71)$$

for $\sigma_{\text{Avg}(P_R)}^2$ and, to good approximation, for $\sigma_{\text{Avg}(P_R)}$. Thus, a formal adjustment of the confidence interval for $\text{Avg}(P_R)$ that accounts for the finiteness of the sample space (maximum number of independent stir states) follows by replacing N with N' in (17). Corresponding definitions for N' can be made for the standard error and confidence intervals for $\text{Std}(P_R)$, etc.

Figure 6 shows the scaling factor N'/N as a function of N for selected values of $N_{\max}$. The graph implies that $\sigma_{\text{Avg}(P_R)}$ for a finite-sized ensemble becomes significantly smaller than $\sigma_{P_R}/\sqrt{N}$ valid for infinite-sized ensembles (viz., by a factor $\sqrt{N/N'}$) when N approaches $N_{\max}$. This results in a lower uncertainty for the sample mean, which may be interpreted as an apparent increase of the sample size N by a factor N'/N . Thus, operating a MT/MSRC close to its limit value $N_{\max}$ (as governed by stirrer characteristics and cavity shape, size and geometry), in particular when $N_{\max}$ is not exceedingly large, is beneficial in reducing the IFU, as this tends to make the random process closer to becoming quasi-deterministic. This is *a fortiori* the case in a MSRC, which traverses and dwells at all $N_{\max}$ stir states, equally. The effect of finite $N_{\max}$ on the confidence interval limits will be analyzed and discussed in Sec. 4.2.2.

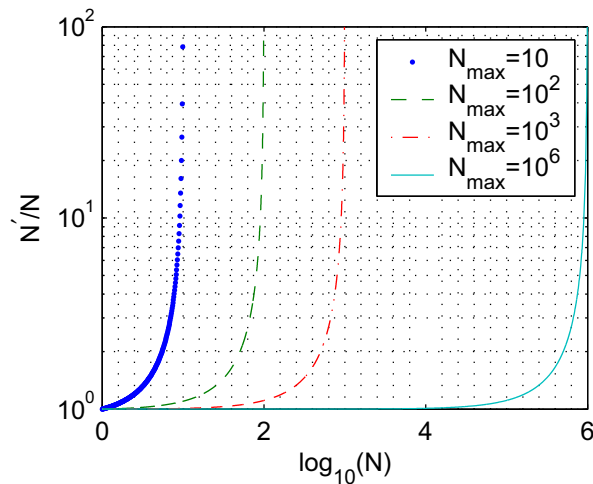


Figure 6: Scaling factor $N'/N = (N_{\max} - 1)/(N_{\max} - N)$ for use with sample variance of P_R and standard error of $\text{Avg}(P_R)$, for subsample sets of N stir states taken from finite-sized ensembles of size $N_{\max} < +\infty$.

4 Chamber Validation and Emissions

4.1 Standard Deviation of Insertion Loss, Received Power and Emitted Power

4.1.1 Substitution Method Based on Prior Estimation of Q-Factor

As already noted in Sec. 3.1.1, the field inside an ideal MT/MSRC is fully incoherent, for electrically large distances between the source and receiving antenna where any contributions by line-of-sight or diffracted fields can be neglected. Hence, the fundamental quantity is the power (or energy) density, integrated over all spatial directions of propagation and polarization. Occasionally, interest is in the field strength (whether its average or maximum value), which is a derived quantity.

Using a substitution method [27], the unknown average value and standard deviation of the total power emitted by an EUT, $\langle P_T \rangle$ and σ_{P_T} , can be estimated from its relation to the average and standard deviation of the received power, $\langle P_R \rangle$ and σ_{P_R} . At any stir state ℓ , the ratio of the insertion loss for the MT/MSRC with the EUT in place, i.e., $P_R(\ell)/P_T(\ell)$, to the insertion loss with the EUT removed from the MT/MSRC (but with the transmitting-receiving antennas in place), i.e., $P_{R,0}(\ell)/P_{T,0}(\ell)$ defines the instantaneous chamber loading factor:

$$\text{CLF}(\ell = \ell_0) \triangleq \frac{P_R(\ell)/P_T(\ell)}{P_{R,0}(\ell_0)/P_{T,0}(\ell_0)}. \quad (72)$$

In (72), the subscript “0” refers to the unloaded MT/MSRC; $P_{T,0}(\ell_0)$ is the power injected into the unloaded chamber by a reference power source, for one of N_0 tuner or stirrer positions; $P_{R,0}(\ell_0)$ is the averaged received power measured in the unloaded chamber for the corresponding position; and $P_R(\ell)$ is the power received in the chamber with the EUT switched on, for one of N stir states (typically, $N \leq N_0$). For good reverberant properties to be preserved, we require $\text{CLF}(\ell_0) \simeq 1, \forall \ell_0$.

To estimate the average radiated (transmitted) power by the EUT, we can consider P_T as a function of P_R , $P_{R,0}$, $P_{T,0}$ and CLF in (72) and expand the resulting expression in a Taylor series, evaluated at $(\langle P_R \rangle, \langle P_{R,0} \rangle, \langle P_{T,0} \rangle, \langle \text{CLF} \rangle)$ and limited to its first two terms²⁵, yielding

$$\langle P_T \rangle \simeq \frac{1}{\langle \text{CLF} \rangle} \frac{\langle P_{T,0} \rangle}{\langle P_{R,0} \rangle} \langle P_R \rangle \simeq \frac{\langle P_{T,0} \rangle}{\langle P_{R,0} \rangle} \langle P_R \rangle \quad (73)$$

where $\langle \text{CLF} \rangle$ is “the” chamber loading factor²⁶ [2, Annex B]. We shall further assume *statistical*

²⁵Strictly, for this first-order approximation to be valid require the fluctuations of P_R , $P_{R,0}$, and $P_{T,0}$ to be small with respect to $\langle P_R \rangle$, $\langle P_{R,0} \rangle$, and $\langle P_{T,0} \rangle$. At least for P_R , $P_{R,0}$, this is only approximately true.

²⁶ $\langle \text{CLF} \rangle (\leq 1)$ accounts for the possible reduction of the Q of the chamber by the presence of the EUT. This factor should remain sufficiently close to one in order to preserve the reverberant properties of the chamber that prevail during chamber validation. For an emitting EUT, evaluation of the switched-off state is necessary in order to avoid (partial or complete) compensation of the deterioration of reverberation quality (through reduction of working volume and/or increased absorption) by an increase of the power generated inside the chamber by the active EUT.

unloading of the MT/MSRC including the EUT, i.e., $\langle \text{CLF} \rangle = 1$ and $\sigma_{\text{CLF}} \ll \langle \text{CLF} \rangle$.

Replacing $\langle P_{\text{T},0} \rangle / \langle P_{\text{R},0} \rangle$ in (73) by $\langle P_{\text{T},0} / P_{\text{R},0} \rangle$ is only valid within this first-order approximation, but can be appealing from a practical point of view, e.g., when using a network analyzer without directional coupler. It does not produce any significant discrepancies provided $N_0 \gg 1$.

For large values of N_0 ($N_0 > 30$), a 95 % confidence interval for $\text{Avg}(P_{\text{T}})$ corresponding to a particular sample average value $\text{avg}(P_{\text{R}})$ is

$$[\text{avg}(P_{\text{T}}) - 1.960 s_{\text{Avg}(P_{\text{T}})}, \text{avg}(P_{\text{T}}) + 1.960 s_{\text{Avg}(P_{\text{T}})}] \equiv \left[\text{avg}(P_{\text{T}}) - 1.960 \frac{s_{P_{\text{T}}}}{\sqrt{N_0}}, \text{avg}(P_{\text{T}}) + 1.960 \frac{s_{P_{\text{T}}}}{\sqrt{N_0}} \right] \quad (74)$$

$$\simeq \left[\langle P_{\text{T}} \rangle - 1.960 \frac{\sigma_{P_{\text{T}}}}{\sqrt{N_0}}, \langle P_{\text{T}} \rangle + 1.960 \frac{\sigma_{P_{\text{T}}}}{\sqrt{N_0}} \right] \quad (75)$$

with [27]

$$\text{avg}(P_{\text{T}}) \simeq \frac{\eta_{\text{R}}}{\eta_{\text{T},0}\eta_{\text{R},0}} \frac{k V}{Q A_{\text{e}}} \frac{\text{avg}(P_{\text{R}})}{\text{CVF}} \simeq \frac{\eta_{\text{R}}}{\eta_{\text{T},0} \eta_{\text{R},0}} \frac{\text{avg}(P_{\text{R}})}{\text{CVF} \text{ avg}(|S_{21,0}|^2)} \quad (76)$$

$$s_{\text{Avg}(P_{\text{T}})} \simeq \frac{\eta_{\text{R}}}{\eta_{\text{T},0}\eta_{\text{R},0}} \frac{k V}{Q A_{\text{e}}} \frac{s_{\text{Avg}(P_{\text{R}})}}{\text{CVF}} \simeq \frac{\eta_{\text{R}}}{\eta_{\text{T},0} \eta_{\text{R},0}} \frac{s_{\text{Avg}(P_{\text{R}})}}{\text{CVF} \text{ avg}(|S_{21,0}|^2)} \quad (77)$$

where $k \triangleq 2\pi/\lambda$ is the wavenumber at wavelength λ ; $A_{\text{e}} = \lambda^2/(8\pi)$ is the antenna effective aperture at λ ; Q is the ensemble averaged quality factor of the chamber at λ ; $\eta_{(\text{R})(\text{T}),0}$ is the radiation efficiency of the receiving or transmitting antenna during chamber validation or emission measurement assumed to be constant with respect to stirring; V is the true or estimated working volume of the chamber;

$$\text{avg}(|S_{21,0}|^2) = \frac{\sum_{\ell=1}^{N_0} |(S_{21,0})_{\ell}|^2}{N_0} \quad (78)$$

is the average transmission S-parameter measured during chamber validation; and CVF represents the chamber validation factor (previously called chamber calibration factor [2, Annex B]). We shall further assume that the loading of the chamber by the EUT is negligible (i.e., $\text{CVF} = 1$).

In (76)–(77), the middle expressions hold in overmoded regime, whilst the right-hand expressions are generally valid. In writing (76)–(77) for ensemble averaged quantities²⁷, it is tacitly assumed that the uncertainty of Q – and, where applicable, A_{e} , V and/or k when considered as physically random and/or numerically ill-determined quantities²⁸ – is much smaller than that for $\text{Avg}(P_{\text{R}})$. In general, the expression of the uncertainty of $\text{Avg}(P_{\text{T}})$ can be obtained from the variance formula; cf. (107). In

²⁷This is the thermodynamic approach, implying $N \rightarrow +\infty$, where the second-order effect of fluctuations of dependent parameters on the average value of the quantity of interest, $\text{Avg}(P_{\text{T}})$, is neglected. This will be examined in Part II.

²⁸Ideally, V should remain fixed during the stirring process, in order to maintain a constant average mode density and, hence, preserve the statistical equivalence of all stir states. The value of V is often only approximately known, due to the complexity of the cavity shape and the boundary zone near antennas and EUT that must be excluded from the interior volume. The central frequency of the instantaneous excitation band fluctuates with varying stir state [28] and can be calculated from a Gabor analytic signal representation. When using frequency stirring (in immunity testing), the frequency sweep must be sufficiently small in order to maintain the average mode density. The antenna radiation

a first approximation, the fluctuations of Q , k , A_e and $\text{Avg}(P_R)$ as the sources of randomness can be assumed to be mutually independent, whence an a priori estimate of $\sigma_{\text{Avg}(P_T)}$ is²⁹ [10, Sec. 5.1.6], [19]

$$s_{\text{Avg}(P_T)} \simeq \text{avg}(P_T) \sqrt{\left[\frac{s_{\text{Avg}(P_R)}}{\text{avg}(P_R)}\right]^2 + \left[\frac{s_Q}{\text{avg}(Q)}\right]^2 + \left[\frac{s_{A_e}}{\text{avg}(A_e)}\right]^2 + \left[\frac{s_V}{\text{avg}(V)}\right]^2 + \left[\frac{s_k}{\text{avg}(k)}\right]^2} \quad (79)$$

where the latter term can usually be neglected in narrowband (CW) excitation. If any correlation³⁰ between these quantities is taken into account, but where the correlation coefficient between each pair of variates is unknown, then an upper bound for the uncertainty $s_{\text{Avg}(P_T)}$ is given by [10, Sec. 5.2.1 Note 1]

$$s_{\text{Avg}(P_T)} \leq \text{avg}(P_T) \left[\frac{s_{\text{Avg}(P_R)}}{\text{avg}(P_R)} + \frac{s_Q}{\text{avg}(Q)} + \frac{s_{A_e}}{\text{avg}(A_e)} + \frac{s_V}{\text{avg}(V)} + \frac{s_k}{\text{avg}(k)} \right] \quad (80)$$

where equality is attained when all quantities are fully correlated.

If only the fluctuations of P_R and Q are not negligibly small, then (79) becomes

$$s_{\text{Avg}(P_T)} \simeq \text{avg}(P_T) \sqrt{\left(\frac{s_{\text{Avg}(P_R)}}{\text{avg}(P_R)}\right)^2 + \left[\frac{s_{\text{Avg}(Q)}}{\text{avg}(Q)}\right]^2} \simeq \sqrt{s_{\text{Avg}(P_R)}^2 \left[\frac{16\pi^2 V}{\lambda^3 \eta_R \text{avg}(Q)}\right]^2 + \left[\frac{s_{\text{Avg}(Q)}}{\text{avg}(Q)}\right]^2 [\text{avg}(P_T)]^2} \quad (81)$$

where the sample statistics of Q can be estimated from M individual stir sequences (obtained with different stir mechanisms or pairs of source/receiver locations space at least $\lambda/2$ apart), and where

$$\text{avg}(Q) = \frac{\sum_{i=1}^M Q_i}{M} = \frac{16\pi^2 V}{M \eta_R \lambda^3} \sum_{i=1}^M \frac{P_{R,i}}{P_{T,i}} \simeq \langle Q \rangle \quad (82)$$

$$s_{\text{Avg}(Q)} = \frac{s_Q}{\sqrt{M}} \simeq \frac{\sigma_Q}{\sqrt{M}} = \sqrt{\frac{\sum_{i=1}^M [Q_i - \langle Q \rangle]^2}{M^2}} \simeq \sqrt{\frac{\sum_{i=1}^M [Q_i - \text{avg}(Q)]^2}{M(M-1)}}. \quad (83)$$

Correspondingly, $\text{avg}(P_R)$ is now calculated based on $N \times L$ measured values during chamber validation, i.e., N and $N-1$ in (1)–(3) are now replaced³¹ by $N \times M$ and $(N-1) \times (M-1)$, respectively ($N, M > 1$).

In Part II, it will be shown that theoretical sampling statistics of $\langle \text{Avg}(Q) \rangle$ and $\sigma_{\text{Avg}(Q)}$ for an ideal chamber are

$$\langle \text{Avg}(Q) \rangle = \langle Q \rangle = \frac{3 \mu_0 V}{2 \mu_w S \delta_w} \frac{w}{w-1} \quad (84)$$

efficiency can usually be assumed to remain constant during the stirring process, even though is often only known approximately. Fluctuations of A_e and λ are not independent. Hence their individual uncertainties cannot be combined via a root-sum-square estimate if uncertainty associated with estimates of both these quantities is to be considered simultaneously. Such dependence between random A_e and λ gives rise to a product of contributions, resulting in a logarithmic-normal pdf (as opposed to a normal pdf) for the overall uncertainty.

²⁹Strictly, (79) is sufficiently accurate only for relatively small respective contributions $s_X/\text{avg}(X)$.

³⁰note that Q , A_e and P_R all depend on λ , i.e., on k , whose uncertainty serves as a source of mutual correlation. However, if we assume $s_k/\text{avg}(k) \ll 1$, as for a case of a narrowband excitation, then the uncertainty of k and correlations due to s_k can be neglected.

³¹If no intermediate averaging over tuner positions at each location is performed, then $N \times M$ is replaced by $N \times (M-1)$.

$$\sigma_{\text{Avg}(Q)} = \frac{\langle Q \rangle}{\sqrt{M}} \sqrt{\frac{(v+1)(w-1)}{v(w-2)} - 1} \simeq \frac{\langle Q \rangle}{\sqrt{M}} \sqrt{\frac{1}{v} + \frac{1}{w}} \propto \frac{1}{M} \quad (85)$$

with

$$v \triangleq 6M \gg 1, \quad w \triangleq \frac{3\lambda S}{2V} M \gg 1 \quad (86)$$

where S is the interior area of all exposed conducting surfaces, which is of the order of $V^{2/3}$; μ_0 and μ_w are the magnetic permeability of free space and of the chamber walls, respectively; and $\delta_w \triangleq 1/\sqrt{\pi\mu_w f}$ is the skin depth of the walls at frequency $f = c/\lambda$. If the stirring process is inefficient, and Q is determined from measurements at a single pair of locations for the transmitting-receiving antennas, then v and w may not be exceedingly large, in which case $\sigma_{\text{Avg}Q}/\langle \text{Avg}(Q) \rangle \not\ll 1$. These expressions for $\langle \text{Avg}(Q) \rangle$ and $\sigma_{\text{Avg}(Q)}$ determine at which point MIU becomes significant in the overall MU budget.

4.1.2 Contribution of Statistical Impedance Mismatch

4.1.2.1 General Considerations In Sec. 4.1.1, the transmitting and receiving antennas were assumed to be impedance matched to the cavity input impedance at the locations of the antennas, for all stir states ℓ . In actuality, at any given ℓ , significant departures from ideal impedance matching to the free-space value may exist, because in overmoded regime only the *ensemble average* of the cavity input impedance over all stir states equals the intrinsic impedance of free space, η_0 . The effect of fluctuations of the impedance mismatch on the uncertainty of $\langle P_T \rangle$ will be investigated next.

Consider the generic test and measurement system depicted in Fig. 7, consisting of a source S, transmitting antenna T, propagation channel (MT/MSRC) for radiated waves, receiving antenna R, detector (analyzer) D, and connecting cables (not shown). For S-parameter measurements in the unloaded chamber, S and D are combined within the same physical device (viz., vector network analyzer); for scalar emissions measurements, S and D are separate devices (viz., EUT and spectrum analyzer, respectively). Since calibration of the measurement system is with reference to the cable ends, the antennas are to be considered as parts or impedance loads of the chamber.

Although the input impedance of an ideal MT/MSRC is on average equal to the free space value η_0 , the input impedance $\eta(\ell)$ at an individual stir state ℓ may significantly depart from this value, as a result of the effect of varying number and level of multiple reflections back to the transmitter. This is expressed by the notion of *statistical impedance match*, defined by $\langle \eta \rangle = \eta_0$ and $\sigma_\eta \neq 0$ or, in terms

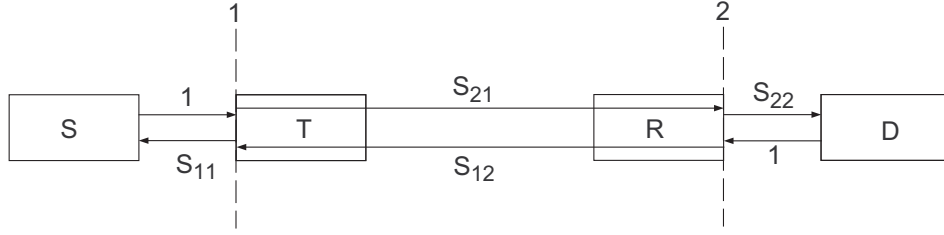


Figure 7: Block diagram of a basic two-port EMC measurement configuration consisting of an RF source (S), transmitting antenna (T), radiation channel between T and R, receiving antenna (R), and detector (D). Cables connect S with T and R with D. Dashed lines represent the phase reference planes. For radiated emissions or immunity testing, S and T combine into the emitting EUT or R and D combine into the susceptible EUT, respectively.

of reflection S-parameters³²,

$$\langle S_{ii} \rangle = 0, \quad \sigma_{S_{ii}}^2 = \langle |S_{ii}|^2 \rangle \neq 0. \quad (i = 1, 2). \quad (87)$$

Deterministic impedance match is included as a special case, i.e., $S_{ii}(\ell) = 0$ and $\sigma_{S_{ii}}^2 = 0$, $\forall \ell$. For the complex electric and magnetic fields at any single stir state, with $E(\ell) = \mathcal{E}[1 + S_{ii}(\ell)]$ and $H(\ell) = [\mathcal{E}/\eta(\ell)][1 - S_{ii}(\ell)]$ where $\mathcal{E}$ is the constant amplitude of the electric field in the absence of mismatch, the ensemble (stir) average of the power launched from the transmitting antenna ($i = 1$) or accepted by the receiving antenna ($i = 2$) is

$$\frac{1}{2} \langle \Re[E(\ell)H^*(\ell)] \rangle = \left\langle \frac{|\mathcal{E}|^2}{2\eta(\ell)} (1 - |S_{ii}(\ell)|^2) \right\rangle \simeq \frac{|\mathcal{E}|^2}{2\eta_0} (1 - \langle |S_{ii}|^2 \rangle) \quad (88)$$

in which the asterisk denotes complex conjugation. Using Mason's rule, the transmission loss P_R/P_T can be related to the insertion loss P_D/P_S via the product³³ of the respective impedance mismatch factors M_S , M_D and antenna efficiencies η_T , η_R : with

$$\frac{V_R^+}{V_S} = \frac{S_{21}}{(1 - S_{11}\Gamma_S)(1 - S_{22}\Gamma_D) - S_{12}S_{21}\Gamma_S\Gamma_D} \quad (89)$$

where V_S^+ represents the voltage at S of a wave travelling in positive direction (i.e., from S to D), we

³²It is emphasized that in a MT/MSRC, the fluctuations of S_{ij} are physically real and inherent to the EME, compared to the classical case where fluctuations of S_{ij} are a result of statistical fluctuations between repeated measurements of a given single state of the EME [13, Sec. D9.6].

³³Sometimes, the antenna efficiency $\eta_{T,R}$ is incorporated into the antenna mismatch factor $1/(1 - |S_{ii}|^2)$, because reflection and ohmic losses have the same overall effect on reducing the net forward transmission. In this case, the redefined power reflection coefficients $|S_{ii}^*|^2$ for use in (91) follow from $\eta_{T,R}(1 - |S_{ii}|^2) = 1 - |S_{ii}^*|^2$ as $|S_{ii}^*|^2 \triangleq 1 - \eta_{T,R}(1 - |S_{ii}|^2)$.

obtain

$$\frac{P_R}{P_T} = \frac{|1 - \Gamma_2 \Gamma_D|^2}{|1 - S_{22} \Gamma_D|^2} \frac{|S_{21}|^2}{\eta_T \eta_R (1 - |\Gamma_1|^2)(1 - |\Gamma_2|^2)} \quad (90)$$

$$\simeq \frac{|S_{21}|^2}{\eta_T (1 - |S_{11}|^2) \eta_R (1 - |S_{22}|^2)} \simeq \frac{P_D/P_S}{\eta_T \eta_R M_S M_D} \quad (91)$$

and, provided $Z_D = Z_S^*$,

$$\frac{P_D}{P_S} = \left| \frac{V_R^+}{V_S} \right|^2 (1 - |\Gamma_S|^2)(1 - |\Gamma_D|^2) = \frac{(1 - |\Gamma_S|^2)(1 - |\Gamma_D|^2)|S_{21}|^2}{|1 - \Gamma_1 \Gamma_S|^2 |1 - S_{22} \Gamma_D|^2} = \frac{(1 - |\Gamma_S|^2)(1 - |\Gamma_D|^2)|S_{21}|^2}{|1 - S_{11} \Gamma_S|^2 |1 - \Gamma_2 \Gamma_D|^2} \quad (92)$$

with

$$P_T = \eta_T (1 - |S_{11}|^2) P_S \quad , \quad P_R = \frac{P_D}{\eta_R (1 - |S_{22}|^2)} \quad , \quad (93)$$

$$M_S = \frac{P_T}{P_S} = \frac{(1 - |\Gamma_S|^2)(1 - |\Gamma_1|^2)}{|1 - \Gamma_1 \Gamma_S|^2} \quad , \quad M_D = \frac{P_D}{P_R} = \frac{(1 - |\Gamma_D|^2)(1 - |\Gamma_2|^2)}{|1 - \Gamma_2 \Gamma_D|^2} \quad , \quad (94)$$

$$\Gamma_1 = S_{11} + \frac{S_{12} S_{21} \Gamma_D}{1 - S_{22} \Gamma_D} \quad , \quad \Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0} \quad , \quad \Gamma_2 = S_{22} + \frac{S_{12} S_{21} \Gamma_S}{1 - S_{11} \Gamma_S} \quad , \quad \Gamma_D = \frac{Z_D - Z_0}{Z_D + Z_0} \quad . \quad (95)$$

In (91), both approximations hold for $|S_{12}| \simeq 0$, i.e., for noninvasive (weakly scattering) receive antennas and the latter approximation holds exactly when $\Gamma_1 = S_{11}$ and $\Gamma_2 = S_{22}$, i.e., when $\Gamma_S = \Gamma_D = 0$ (matched transmitter and receiver). For a reciprocal system, $S_{12} = S_{21}$.

If $\langle |S_{11}|^2 \rangle = \langle |S_{22}|^2 \rangle$, as is the case if both the transmitting and receiving antennas are sufficiently far away (viz., quarter-wavelength distance or farther) from the nearest conducting surface, and neglecting for the moment any antenna inefficiencies, then

$$\left\langle \frac{P_R}{P_T} \right\rangle \simeq \frac{\langle P_R \rangle}{\langle P_T \rangle} = \frac{\langle |S_{21}|^2 \rangle}{(1 - \langle |S_{11}|^2 \rangle)(1 - \langle |S_{22}|^2 \rangle)} = \frac{\langle |S_{21}|^2 \rangle}{(1 - \langle |S_{ii}|^2 \rangle)^2} \quad (96)$$

For a well-stirred chamber, i.e., at operating frequencies well above its lowest usable frequency (LUF) [2, Annex B] and in the absence of direct illumination³⁴ (i.e., no line-of-sight field contribution) for transmission with statistically matched input and output impedances ($\langle S_{ii} \rangle = 0$, $\sigma_{S_{ii}} \neq 0$), the magnitudes of the reflection and transmission parameters have been theoretically shown and experimentally verified to satisfy [29, Sec. 2.1]

$$\sigma_{|S_{21}|}^2 = \frac{\sigma_{|S_{11}|} \sigma_{|S_{22}|}}{2} \quad (97)$$

or, equivalently, in terms of the average intensities [30]

$$\langle |S_{21}|^2 \rangle = \frac{1}{4} \left(\langle |S_{11}|^2 \rangle + \langle |S_{22}|^2 \rangle \right) = \frac{1}{2} \langle |S_{11}|^2 \rangle = \frac{1}{2} \langle |S_{22}|^2 \rangle \quad (98)$$

³⁴The removal of any (deterministic) bias field, so that only (random) fluctuations remain, corresponds to the classical assumption in uncertainty analysis that any systematic uncertainties have been identified and reduced to a negligible level relative to the random uncertainties.

as a manifestation of enhanced backscattering (“glory” effect; cf., e.g., [31]). Indeed, from (98) in combination with the definition

$$\sigma_{S_{ij}}^2 \triangleq \langle S_{ij} S_{ij}^* \rangle - \langle S_{ij} \rangle \langle S_{ij}^* \rangle \equiv \langle |S_{ij}|^2 \rangle - |\langle S_{ij} \rangle|^2, \quad (i, j = 1, 2) \quad (99)$$

together with (87) and

$$\langle |S_{ij}| \rangle = \frac{\sqrt{\pi}}{2} \sqrt{\langle |S_{ij}|^2 \rangle}, \quad (i, j = 1, 2) \quad (100)$$

which is valid for ideal $\chi_2^{(2)}$ distributed (squared) magnitudes of unbiased circular Gauss normal S_{ij} , [15], it is easily verified that (97) and (98) are equal and that, moreover, the following identities result:

$$\sigma_{S_{21}}^2 = \frac{\sigma_{S_{11}} \sigma_{S_{22}}}{2}, \quad \sigma_{S_{ij}}^2 = \frac{4}{4 - \pi} \sigma_{|S_{ij}|}^2, \quad (i, j = 1, 2) \quad (101)$$

$$\sigma_{|S_{21}|^2}^2 = \sigma_{S_{21}}^4 = \frac{\sigma_{|S_{11}|^2} \sigma_{|S_{22}|^2}}{4}, \quad \sigma_{|S_{ij}|^2} \equiv \sigma_{1-|S_{ij}|^2} = \sigma_{S_{ij}}^2 = \langle |S_{ij}|^2 \rangle, \quad (i, j = 1, 2) \quad (102)$$

where (102) follows based on $\langle |S_{ij}|^4 \rangle = 2 \langle |S_{ij}|^2 \rangle^2$ for complex-valued S_{ij} . Thus, for an impedance matched source and detector, (96) reduces with (98) to

$$\frac{\langle P_R \rangle}{\langle P_T \rangle} \simeq \frac{\langle |S_{21}|^2 \rangle}{(1 - 2 \langle |S_{21}|^2 \rangle)^2}. \quad (103)$$

The sampling and sample standard deviation of the average transmission loss are then

$$\sigma_{\text{Avg}(|S_{21}|^2)} = \frac{\sigma_{|S_{21}|^2}}{\sqrt{N}} = \frac{\langle |S_{21}|^2 \rangle}{\sqrt{N}}, \quad s_{\text{Avg}(|S_{21}|^2)} \simeq \frac{\text{avg}(|S_{21}|^2)}{\sqrt{N}}. \quad (104)$$

Similarly, for the complement of the mismatch factors of statistically matched transmitter and receiver,

$$\sigma_{1-\text{Avg}(|S_{ii}|^2)} \equiv \sigma_{\text{Avg}(|S_{ii}|^2)} = \frac{\sigma_{|S_{ii}|^2}}{\sqrt{N}} = \frac{\langle |S_{ii}|^2 \rangle}{\sqrt{N}}, \quad s_{1-\text{Avg}(|S_{ii}|^2)} \simeq \frac{\text{avg}(|S_{ii}|^2)}{\sqrt{N}}. \quad (105)$$

With the aid of (101)–(102), $\sigma_{\text{Avg}(|S_{21}|^2)}$ and $\sigma_{1-\text{Avg}(|S_{ii}|^2)}$ can be expressed in terms of the standard deviation of the complex S-parameters themselves or their magnitude. Furthermore, from (101), (104), and (105), provided $\sigma_{S_{11}} = \sigma_{S_{22}}$,

$$s_{1-\text{Avg}(|S_{ii}|^2)} = 2 s_{\text{Avg}(|S_{21}|^2)}. \quad (106)$$

4.1.2.2 Chamber Validation The quantities of primary interest during chamber validation are the insertion loss P_R/P_T and the received power P_R (or received field strength). We shall introduce an additional subscript “0”, as in e.g. $|S_{11,0}|^2$, to explicitly indicate that a quantity refers to a reference value, measured during chamber validation (i.e., in the “empty” chamber, without EUT in place), as opposed to a loaded chamber.

The standard deviation of $P_{R,0}/P_{T,0}$ or $\text{Avg}(P_{R,0}/P_{T,0})$ can be obtained from the variance formula. For $\sigma_{P_{R,0}/P_{T,0}}$, denoting (91) formally as $X = X_{21}/(X_{11}X_{22})$, this formula reads

$$\begin{aligned} \sigma_X^2 &= \left(\frac{\partial X}{\partial X_{11}} \right)^2 \Big|_{X=x} \sigma_{X_{11}}^2 + \left(\frac{\partial X}{\partial X_{22}} \right)^2 \Big|_{X=x} \sigma_{X_{22}}^2 + \left(\frac{\partial X}{\partial X_{21}} \right)^2 \Big|_{X=x} \sigma_{X_{21}}^2 + 2 \left(\frac{\partial X}{\partial X_{11}} \right) \Big|_{X=x} \left(\frac{\partial X}{\partial X_{22}} \right) \Big|_{X=x} \sigma_{X_{11},X_{22}} \\ &\quad + 2 \left(\frac{\partial X}{\partial X_{11}} \right) \Big|_{X=x} \left(\frac{\partial X}{\partial X_{21}} \right) \Big|_{X=x} \sigma_{X_{11},X_{21}} + 2 \left(\frac{\partial X}{\partial X_{22}} \right) \Big|_{X=x} \left(\frac{\partial X}{\partial X_{21}} \right) \Big|_{X=x} \sigma_{X_{22},X_{21}} \end{aligned} \quad (107)$$

$$\leq \left(\left| \frac{\partial X}{\partial X_{11}} \right|_{X=x} \sigma_{X_{11}} + \left| \frac{\partial X}{\partial X_{22}} \right|_{X=x} \sigma_{X_{22}} + \left| \frac{\partial X}{\partial X_{21}} \right|_{X=x} \sigma_{X_{21}} \right)^2. \quad (108)$$

For $X = X_{21}/(X_{11}X_{22})$, with $|\rho_{X_{ij},X_{i'j'}}| \equiv |\sigma_{X_{ij},X_{i'j'}}|/(\sigma_{X_{ij}}\sigma_{X_{i'j'}}) \leq 1$ and $\text{sgn}(\rho_{X_{11},X_{21}}) \times \text{sgn}(\rho_{X_{22},X_{21}}) = \text{sgn}(\rho_{X_{11},X_{22}})$, this can be written as

$$\nu_X^2 \equiv \frac{\sigma_X^2}{\langle X \rangle^2} = \nu_{X_{11}}^2 + \nu_{X_{22}}^2 + \nu_{X_{21}}^2 + 2\nu_{X_{11}}\nu_{X_{22}}\rho_{X_{11},X_{22}} - 2\nu_{X_{11}}\nu_{X_{21}}\rho_{X_{11},X_{21}} - 2\nu_{X_{22}}\nu_{X_{21}}\rho_{X_{22},X_{21}} \quad (109)$$

$$\leq (\nu_{X_{11}} + \nu_{X_{22}} + \nu_{X_{21}})^2, \quad (110)$$

with equality in (110) being reached, in principle, when $\rho_{X_{11},X_{21}} = \rho_{X_{22},X_{21}} = -\rho_{X_{11},X_{22}} = -1$. However, for a general reciprocal two-port network, which serves as an approximation for a high- Q MT/MSRC not containing nonreciprocal materials or devices, the identities³⁵ $|S_{11,0}|^2 + |S_{21,0}|^2 \leq 1$ and $|S_{12,0}|^2 + |S_{22,0}|^2 = |S_{21,0}|^2 + |S_{22,0}|^2 \leq 1$ hold, i.e., $|S_{21,0}|^2$, $1 - |S_{11,0}|^2$ and $1 - |S_{22,0}|^2$ are positively correlated³⁶ via linear relationships, so that all three correlation coefficients $\rho_{X_{ij},X_{i'j'}}$ equal ideally +1 for unbiased ideal random fields. In any case, (107) with (98), (102) and $\nu_{|S_{21,0}|^2} = 1$ yields

$$\begin{aligned} \left(\frac{\sigma_{P_{R,0}/P_{T,0}}}{\langle P_{R,0}/P_{T,0} \rangle} \right)^2 &= \left(\frac{\sigma_{1-|S_{11,0}|^2}}{1 - \langle |S_{11,0}|^2 \rangle} \right)^2 + \left(\frac{\sigma_{1-|S_{22,0}|^2}}{1 - \langle |S_{22,0}|^2 \rangle} \right)^2 + \left(\frac{\sigma_{|S_{21,0}|^2}}{\langle |S_{21,0}|^2 \rangle} \right)^2 + 2 \frac{\sigma_{1-|S_{11,0}|^2, 1-|S_{22,0}|^2}}{(1 - \langle |S_{11,0}|^2 \rangle)(1 - \langle |S_{22,0}|^2 \rangle)} \\ &\quad - 2 \frac{\sigma_{1-|S_{11,0}|^2, |S_{21,0}|^2}}{(1 - \langle |S_{11,0}|^2 \rangle) \langle |S_{21,0}|^2 \rangle} - 2 \frac{\sigma_{1-|S_{22,0}|^2, |S_{21,0}|^2}}{(1 - \langle |S_{22,0}|^2 \rangle) \langle |S_{21,0}|^2 \rangle} \end{aligned} \quad (111)$$

$$\begin{aligned} &= 1 - 2(\rho_{|S_{21,0}|^2, 1-|S_{11,0}|^2} + \rho_{|S_{21,0}|^2, 1-|S_{22,0}|^2}) \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) \\ &\quad + 2(1 + \rho_{|S_{11,0}|^2, |S_{22,0}|^2}) \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2 \end{aligned} \quad (112)$$

With all correlation functions equalling +1, this becomes

$$\left(\frac{\sigma_{P_{R,0}/P_{T,0}}}{\langle P_{R,0}/P_{T,0} \rangle} \right)^2 = 1 - 4 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) + 4 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2. \quad (113)$$

The influence of $\langle |S_{21,0}|^2 \rangle$ becomes substantial if $\langle |S_{21,0}|^2 \rangle \ll 1/10$ (~ -10 dB). Provided $\langle |S_{21,0}|^2 \rangle \ll 1/2$, as is typically the case in practical MT/MSRCs, this simplifies to

$$\nu_{P_{R,0}/P_{T,0}} \simeq \sqrt{1 - 8\langle |S_{21,0}|^2 \rangle} \simeq 1 - 4\langle |S_{21,0}|^2 \rangle. \quad (114)$$

³⁵In our case of wireless transmission, only a fraction of the transmitted power is received by the antenna due to radiation loss, at any given ℓ , with the rest of the power being lost to reception but serving to generate the cavity field and build the statistically homogeneous energy density at other locations (this local reception is statistical rather than deterministic). Thus, $|S_{11,0}|^2 + |S_{21,0}|^2 < 1$ and $|S_{21,0}|^2 + |S_{22,0}|^2 < 1$ (in practice, $\ll 1$). However, the proportionality of $|S_{21,0}|^2$ to $1 - |S_{11,0}|^2$ and $1 - |S_{22,0}|^2$ still holds because of energy conservation.

³⁶The correlations exist provided all four S-parameters are measured simultaneously at each ℓ . If these parameters are measured during different tuner or stirrer sweeps, no correlations at ℓ can be assumed; in this case, (112) and (114) become $\nu_{P_{R,0}/P_{T,0}} = \sqrt{1 + 2[2\langle |S_{21,0}|^2 \rangle / (1 - 2\langle |S_{21,0}|^2 \rangle)]^2} \simeq \sqrt{1 + 8\langle |S_{21,0}|^2 \rangle^2} \simeq 1 + 4\langle |S_{21,0}|^2 \rangle^2$, i.e., a larger uncertainty.

Thus, the effect of instantaneous impedance mismatch on the coefficient of variation of the insertion loss for ideal random fields is a reduction, of first order with respect to the average value (or, equivalently, standard deviation) of the power transmission S-parameter.

In practice, correlations involving $1 - |S_{ii,0}|^2$ tend to be much smaller, because they are sensitive to cabling layout and other perturbations. As an extreme case, the correlation coefficients $\rho_{X_{ij}, X_{i'j'}}$ may be taken as zero. Then, instead of (113) and (114),

$$\nu_{P_{R,0}/P_{T,0}}^2 = 1 + 2 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2, \quad \nu_{P_{R,0}/P_{T,0}} \simeq 1 + 4 \langle |S_{21,0}|^2 \rangle^2, \quad (115)$$

i.e., a second-order *increase* of the relative uncertainty compared to positive full correlation.

Experimental (sample) values $\text{avg}(|S_{ij,0}|^2)$ and $s_{|S_{ij,0}|^2}$ can be substituted for $\langle |S_{ij,0}|^2 \rangle$ and $\sigma_{|S_{ij,0}|^2}$, respectively, to determine sample estimates $n_{P_{R,0}/P_{T,0}}^{(2)}$ for $\nu_{P_{R,0}/P_{T,0}}^{(2)}$.

For the uncertainty of $P_{R,0}$, whose value is determined by both $P_{T,0}$ and chamber properties as

$$P_{R,0} = \frac{|S_{21,0}|^2}{(1 - |S_{11,0}|^2)(1 - |S_{22,0}|^2)} P_{T,0}, \quad (116)$$

we can consider the random fluctuations of $P_{T,0}$ to be uncorrelated with those of $S_{21,0}$ or $S_{22,0}$, while they can be strongly correlated with $S_{11,0}$. The absence of correlations of $P_{T,0}$ with $S_{21,0}$ is a result of the fact that fluctuations of the transmission S-parameter of the MT/MSRC during the stirring process are governed by dynamics of cavity geometry, which is being modified separately from (and is, therefore, unrelated to any³⁷ fluctuations of) the radiating source. On the other hand, the potentially strong correlation of $P_{T,0}$ with fluctuations of $S_{11,0}$ occurs when the nominal source power P_S is maintained constant during the stirring process, whence any fluctuations in the power launched from the transmitting antenna located at $\underline{r}_{T,0}$ are then a direct result of fluctuations of the local instantaneous input impedance $\eta(\underline{r}_{T,0}, \ell)$ due to dynamics of geometry. Finally, since the location of the receive antenna at $\underline{r}_{R,0}$ is separated from the transmit antenna typically by at least several correlation lengths ($k|\underline{r}_{R,0} - \underline{r}_{T,0}| \gg \pi$), the fluctuations of $\eta(\underline{r}_{T,0}, \ell)$ are not correlated to those of $\eta(\underline{r}_{R,0}, \ell)$, whence $P_{T,0}$ and $S_{22,0}$ are uncorrelated. Thus, the relative uncertainty of (116) is in general

$$\begin{aligned} \nu_{P_{R,0}}^2 &= \left(\frac{\sigma_{|S_{11,0}|^2}}{1 - \langle |S_{11,0}|^2 \rangle} \right)^2 + \left(\frac{\sigma_{|S_{22,0}|^2}}{1 - \langle |S_{22,0}|^2 \rangle} \right)^2 + \left(\frac{\sigma_{|S_{21,0}|^2}}{\langle |S_{21,0}|^2 \rangle} \right)^2 + \left(\frac{\sigma_{P_{T,0}}}{\langle P_{T,0} \rangle} \right)^2 \\ &\quad + 2 \frac{\sigma_{|S_{11,0}|^2, |S_{22,0}|^2}}{(1 - \langle |S_{11,0}|^2 \rangle)(1 - \langle |S_{22,0}|^2 \rangle)} - 2 \frac{\sigma_{|S_{21,0}|^2, |S_{11,0}|^2}}{\langle |S_{21,0}|^2 \rangle (1 - \langle |S_{11,0}|^2 \rangle)} \\ &\quad - 2 \frac{\sigma_{|S_{21,0}|^2, |S_{22,0}|^2}}{\langle |S_{21,0}|^2 \rangle (1 - \langle |S_{22,0}|^2 \rangle)} - 2 \frac{\sigma_{|S_{11,0}|^2} \sigma_{P_{T,0}}}{(1 - \langle |S_{11,0}|^2 \rangle) \langle P_{T,0} \rangle} \rho_{1 - |S_{11,0}|^2, P_{T,0}} \end{aligned} \quad (117)$$

$$\begin{aligned} &= 1 + \nu_{P_{T,0}}^2 - 2 (\rho_{|S_{21,0}|^2, 1 - |S_{11,0}|^2} + \rho_{|S_{21,0}|^2, 1 - |S_{22,0}|^2} - \nu_{P_{T,0}} \rho_{1 - |S_{11,0}|^2, P_{T,0}}) \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) \\ &\quad + 2 (1 + \rho_{|S_{11,0}|^2, |S_{22,0}|^2}) \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2 \end{aligned} \quad (118)$$

³⁷If the source power fluctuates by itself, e.g., by excitation with additive white Gaussian noise (AWGN) or for a randomly modulated source signal, this will affect both $\sigma_{P_{T,0}}$ and $\sigma_{P_{R,0}}$, irrespective of the values of $S_{11,0}$ and $S_{22,0}$, whereas the statistical distributions (assumed to be ideal for any excitation) remain constant.

and with all correlation coefficients equalling +1,

$$\nu_{P_{R,0}}^2 = 1 + \nu_{P_{T,0}}^2 + 2(\nu_{P_{T,0}} - 2) \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) + 4 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2. \quad (119)$$

For $\langle |S_{21,0}|^2 \rangle \ll 1$, this simplifies to

$$\nu_{P_{R,0}}^2 \simeq 1 + \nu_{P_{T,0}}^2 + 4(\nu_{P_{T,0}} - 2)\langle |S_{21,0}|^2 \rangle + 16\langle |S_{21,0}|^2 \rangle^2. \quad (120)$$

For negligible fluctuations of the emitted power $P_{T,0}$ ($\sigma_{P_{T,0}} \ll \langle P_{T,0} \rangle$), we retrieve (113)–(114), i.e.,

$$\nu_{P_{R,0}}^2 = 1 - 4 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) + 4 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2 \simeq 1 - 8\langle |S_{21,0}|^2 \rangle. \quad (121)$$

On the other hand, in the absence of correlations,

$$\nu_{P_{R,0}}^2 = 1 + \nu_{P_{T,0}}^2 + 2 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2 \simeq 1 + \nu_{P_{T,0}}^2 + 8\langle |S_{21,0}|^2 \rangle^2 \simeq 1 + 8\langle |S_{21,0}|^2 \rangle^2. \quad (122)$$

Conversely, the uncertainty of $P_{T,0}$ could be estimated based on $P_{R,0}$ from the inverse relation

$$P_{T,0} = \frac{(1 - |S_{11,0}|^2)(1 - |S_{22,0}|^2)}{|S_{21,0}|^2} P_{R,0}. \quad (123)$$

Again using the variance formula (107), now applied to $Y \triangleq P_{R,0}/X$ and noting that the relative uncertainties of X and $1/X$ are the same in this case, i.e., $\nu_{P_{R,0}/P_{T,0}}^2 = \nu_{P_{T,0}/P_{R,0}}^2$, this yields

$$\begin{aligned} \nu_{P_{T,0}}^2 &= 1 - 2\nu_{P_{R,0}}\rho_{|S_{21,0}|^2, P_{R,0}} + \nu_{P_{R,0}}^2 + 2(1 + \rho_{|S_{11,0}|^2, |S_{22,0}|^2}) \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2 \\ &\quad - 2[\rho_{1-|S_{11,0}|^2, |S_{21,0}|^2} + \rho_{1-|S_{22,0}|^2, |S_{21,0}|^2} - \nu_{P_{R,0}}(\rho_{1-|S_{11,0}|^2, P_{R,0}} + \rho_{1-|S_{22,0}|^2, P_{R,0}})] \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right). \end{aligned} \quad (124)$$

With all correlation coefficients equalling +1,

$$\nu_{P_{T,0}}^2 = 1 - 2\nu_{P_{R,0}} + \nu_{P_{R,0}}^2 + 4(\nu_{P_{R,0}} - 1) \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) + 4 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2. \quad (125)$$

For $\langle |S_{21,0}|^2 \rangle \ll 1$, this simplifies to

$$\nu_{P_{T,0}}^2 \simeq 1 - 2\nu_{P_{R,0}} + \nu_{P_{R,0}}^2 + 8(\nu_{P_{R,0}} - 1)\langle |S_{21,0}|^2 \rangle + 16\langle |S_{21,0}|^2 \rangle^2. \quad (126)$$

Ideally, for an exponentially distributed $P_{R,0}$, we have that $\nu_{P_{R,0}} = 1$ and

$$\nu_{P_{T,0}} = 2 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) \simeq 4\langle |S_{21,0}|^2 \rangle. \quad (127)$$

In the absence of correlations, we obtain of course the equivalent of (122), i.e.,

$$\nu_{P_{T,0}}^2 = 1 + \nu_{P_{R,0}}^2 + 2 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2 \simeq 1 + \nu_{P_{R,0}}^2 + 8\langle |S_{21,0}|^2 \rangle^2. \quad (128)$$

It must be emphasized that the accuracy of estimation of IFU that is based on (107) is inevitably limited. More accurate expressions of IFU require characterization and use of pdfs. This will be investigated in Part II.

4.1.2.3 Emissions For the substitution method (72), the relative uncertainty of the transmitted power of the emitting source is

$$\nu_{P_T}^2 = \nu_{P_R}^2 + \nu_{P_{T,0}/P_{R,0}}^2 + 2 \nu_{P_R} \nu_{P_{T,0}/P_{R,0}} \rho_{P_R, P_{T,0}/P_{R,0}}. \quad (129)$$

Since emission and chamber validation measurements involve different measurements, $\rho_{P_R, P_{T,0}/P_{R,0}} = 0$ by definition. Hence, with $\nu_{P_{T,0}/P_{R,0}}^2 = \nu_{P_{R,0}/P_{T,0}}^2$ given by (113), we obtain

$$\nu_{P_T}^2 = \nu_{P_R}^2 + 1 - 4 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) + 4 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2. \quad (130)$$

Generally, the power P_T radiated (emitted) by an EUT can be calculated from the detected power P_D (as fed by the receiving antenna to the analyzer) as

$$P_T = \frac{(1 - |\Gamma_1|^2)(1 - |S_{22}\Gamma_D|^2)}{|S_{21}|^2(1 - |\Gamma_D|^2)} P_D = \frac{1 - |S_{11}|^2}{|S_{21}|^2} P_D \quad (131)$$

$$= \frac{\eta_{T,0}\eta_{R,0}}{\eta_R} \frac{(1 - |S_{11,0}|^2)(1 - |S_{22,0}|^2)}{(1 - |S_{22}|^2)} \frac{P_D}{|S_{21,0}|^2} \quad (132)$$

where the last equality in (131) holds in case of a matched detector ($\Gamma_D = 0$, i.e., $\Gamma_1 = S_{11}$), as we shall further assume. In an ideal chamber ($\nu_{|S_{21}|^2} = 1$) with an ideal power meter or analyzer ($\nu_{P_D} = \nu_{|S_{21}|^2}$) and with the same or a similarly matched receiving antenna or power sensor compared to the one used during chamber validation ($\nu_{1-|S_{22}|^2}^2 = \nu_{1-|S_{22,0}|^2}^2$), for an EUT that does not significantly load the chamber ($\text{CLF} = 1$), we obtain with the aid of (93)

$$\nu_{P_R}^2 = \nu_{P_D}^2 + \nu_{1-|S_{22}|^2}^2 = \nu_{P_D}^2 + \left(\frac{1}{1 - 2\langle |S_{21}|^2 \rangle} - 1 \right)^2 \simeq 1 + 4 \langle |S_{21,0}|^2 \rangle^2 \quad (133)$$

whence a theoretical estimate of $\nu_{P_T}^2$ follows from (130) and (133) as

$$\nu_{P_T}^2 = \nu_{P_D}^2 + \nu_{1-|S_{22}|^2}^2 + \nu_{P_{T,0}/P_{R,0}}^2 \quad (134)$$

$$\simeq \nu_{P_D}^2 + 1 - 4 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) + 5 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2 \simeq 2 - 8\langle |S_{21,0}|^2 \rangle. \quad (135)$$

Asymptotically, for $\langle |S_{21,0}(f \rightarrow +\infty)|^2 \rangle \rightarrow 0$, we have $\nu_{P_T} = 2$ (~ 1.5 dB). For $\nu_{P_D}^2 \simeq 1$, the influence of $\langle |S_{21,0}|^2 \rangle$ becomes significant when $\langle |S_{21,0}|^2 \rangle \ll 1/6$ (~ -7.8 dB). On the other hand, if all correlation coefficients for the chamber validation are (or set) zero, then with (115) instead of (113), we have

$$\nu_{P_T}^2 \simeq \nu_{P_D}^2 + 1 + 3 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2 \simeq 2 + 12\langle |S_{21,0}|^2 \rangle^2 \quad (136)$$

instead of (135).

For intra- and intercomparisons of EUT emission levels in nonreverberant or other reverberant EMEs, respectively, it may be of interest to determine the internal source power P_S instead of P_T ,

because P_S is independent of the EME. This source power can be estimated based from the emitted power using (93), i.e.,

$$P_S = \frac{P_T}{\eta_T(1 - |S_{11}|^2)} \quad (137)$$

and P_S as a function of P_D given by (92) or, in the substitution method and provided $\Gamma_D = 0$,

$$P_S = \frac{\eta_{T,0} (1 - |S_{11,0}|^2) \eta_{R,0} (1 - |S_{22,0}|^2)}{\eta_T(1 - |S_{11}|^2) \eta_R (1 - |S_{22}|^2) |S_{21,0}|^2} P_D. \quad (138)$$

With (93) and $\rho_{1-|S_{11}|^2, 1-|S_{22}|^2} = +1$, the relative uncertainty of P_S can be estimated from

$$\nu_{P_S}^2 = \nu_{P_T}^2 + \nu_{1-|S_{11}|^2}^2 - 2\nu_{P_T} \nu_{1-|S_{11}|^2} \rho_{P_T, 1-|S_{11}|^2} \quad (139)$$

$$= \nu_{P_D}^2 + \nu_{1-|S_{22}|^2}^2 + \nu_{1-|S_{11}|^2}^2 + \nu_{P_{T,0}/P_{R,0}}^2 - 2\nu_{1-|S_{11}|^2} \nu_{1-|S_{22}|^2} \rho_{1-|S_{11}|^2, 1-|S_{22}|^2} \quad (140)$$

$$= \nu_{P_D}^2 + 1 - (4 + 2\nu_{1-|S_{11}|^2}) \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right) + 5 \left(\frac{1}{1 - 2\langle |S_{21,0}|^2 \rangle} - 1 \right)^2 + \nu_{1-|S_{11}|^2}^2. \quad (141)$$

Unfortunately, $\nu_{1-|S_{11}|^2}$ is difficult to estimate accurately for unintentional radiators, which are typically strongly mismatched to any reference or other EME, whence an estimate as $\nu_{1-|S_{11,0}|^2}$ may be poor.

The effect of the on-average mismatch between the output impedance of an EUT and the input impedance of the cavity at an arbitrary stir state can be analyzed in more detail. In Part II, it will be shown how $\nu_{1-|S_{11}|^2} = \sigma_{1-|S_{11}|^2} / (1 - \langle |S_{11}|^2 \rangle)$ can be calculated from the pdf of $|S_{11}|^2$. The dependence of $\nu_{1-|S_{11}|^2}^2$ on the on-average mismatch (nonzero $\langle |S_{11}|^2 \rangle$) is depicted in Fig. 8 for selected values of the spread of the fluctuations of the mismatch, $\sigma_{S_{11}}$. In effect, the case $\sigma_{S_{11}} \rightarrow +\infty$ corresponds to negligible on-average mismatch. It can be seen that, unless $\sigma_{S_{11}}$ is not small in absolute terms, the mismatch can give rise to a substantial increase of $\nu_{1-|S_{11}|^2}$ and, hence, an increased uncertainty for the radiated power.

4.1.2.4 Standard Error of Sample Mean Power For $\nu_{\text{Avg}(X)}^{(2)}$, the expressions in (113), (114), etc. are to be multiplied by $1/N$ and $1/\sqrt{N}$, respectively. This follows from the fact that $\sigma_{\text{Avg}(X)}^2$ is linear in $1/N$: for example, with $\text{Avg}(X) = \sum_{i=1}^N X_i/N$, the first term in (107) becomes

$$\left(\frac{\partial \text{Avg}(X)}{\partial X_{11}} \right)^2 \bigg|_{\text{Avg}(X)=\text{avg}(X)} \sigma_{X_{11}}^2 = \sum_{i=1}^N \left(\frac{\partial \text{Avg}(X)}{\partial X_i} \frac{\partial X_i}{\partial X_{11,i}} \right)^2 \bigg|_{\text{Avg}(X)=\text{avg}(X)} \sigma_{X_{11}}^2 \quad (142)$$

$$= \sum_{i=1}^N \left(\frac{1}{N} \frac{\partial X}{\partial X_{11}} \right)^2 \bigg|_{X=x} \sigma_{X_{11}}^2 = \frac{1}{N} \left(\frac{\partial X}{\partial X_{11}} \right)^2 \bigg|_{X=x} \sigma_{X_{11}}^2. \quad (143)$$

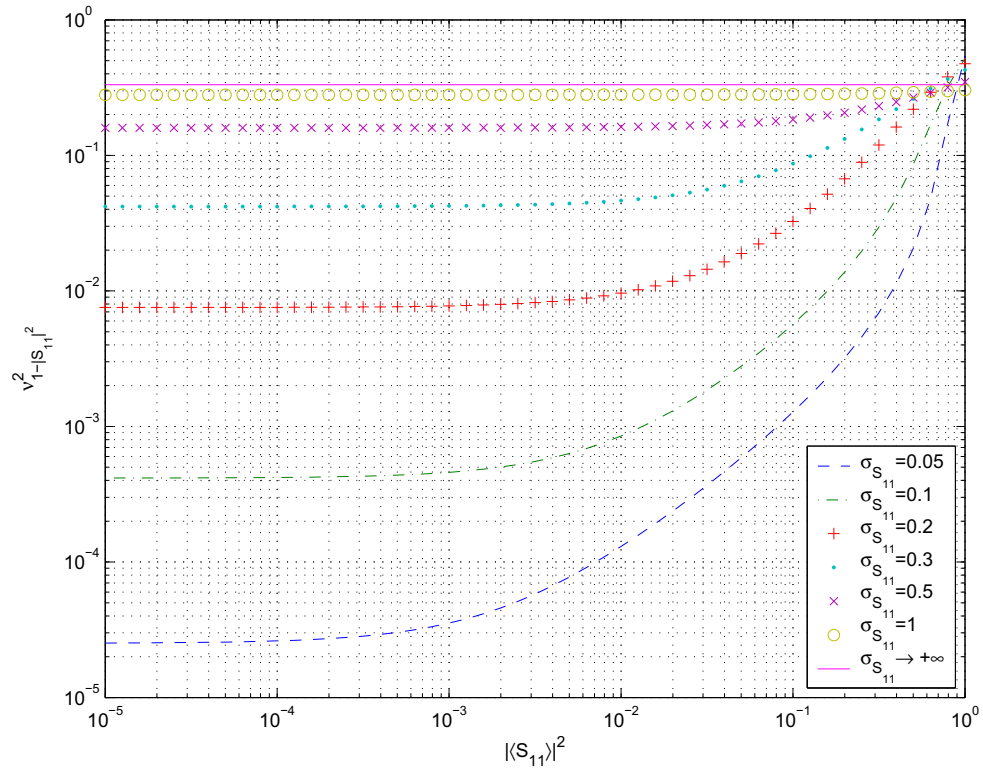


Figure 8: Dependence of $\nu_{1-|S_{11}|^2}^2$ on the average impedance mismatch $|S_{11}|^2$ (deterministic) of an emitting EUT, for selected values of the spread of mismatch fluctuations $\sigma_{S_{11}}$.

4.2 Closed-Form Expressions For Confidence Intervals for Average and Standard Deviation of Power and Field Strength

4.2.1 Ideal Chamber ($N_{\max} \rightarrow +\infty$)

4.2.1.1 Expected Value The previous results for the standard deviation of the average power or field strength can be used to propose analytical expressions for $\eta\%$ confidence intervals, to yield practically useful estimates. To this end, the appropriate coefficient of variation needs to be used, to replace the value $\sigma_{P_{R(T),\alpha}}/\langle P_{R(T),\alpha} \rangle = 1$ for the χ_2^2 distributed power for a Cartesian field component α [16]. Exact results based on χ_{2pN}^2 distributions were obtained with the results from Sec. 3.1.1. For practical use, we here list 95 % confidence intervals that are applicable for $N > 30$ and $N_{\max} \rightarrow +\infty$, for the average of the Cartesian and vector received power or field strength.

- for the Cartesian power $P_{R,\alpha}$: with $\sigma_{P_{R,\alpha}}/\langle P_{R,\alpha} \rangle = 1$,

$$\text{avg}(P_{R,\alpha}) \left(1 - \frac{1.960}{\sqrt{N}} \right) \leq \langle P_{R,\alpha} \rangle \leq \text{avg}(P_{R,\alpha}) \left(1 + \frac{1.960}{\sqrt{N}} \right); \quad (144)$$

- for the vectorial power $P_{R,t}$: with $\sigma_{P_{R,t}}/\langle P_{R,t} \rangle = 1/\sqrt{3}$,

$$\text{avg}(P_{R,t}) \left(1 - \frac{1.960}{\sqrt{3}N}\right) \leq \langle P_{R,t} \rangle \leq \text{avg}(P_{R,t}) \left(1 + \frac{1.960}{\sqrt{3}N}\right); \quad (145)$$

- for the Cartesian field strength $|E_{R,\alpha}|$: with $\sigma_{|E_{R,\alpha}|}/\langle |E_{R,\alpha}| \rangle = \sqrt{(4/\pi) - 1}$,

$$\text{avg}(|E_{R,\alpha}|) \left(1 - 1.960\sqrt{\frac{4-\pi}{\pi N}}\right) \leq \langle |E_{R,\alpha}| \rangle \leq \text{avg}(|E_{R,\alpha}|) \left(1 + 1.960\sqrt{\frac{4-\pi}{\pi N}}\right); \quad (146)$$

- for the vectorial field strength $|E_{R,t}|$: with $\sigma_{|E_{R,t}|}/\langle |E_{R,t}| \rangle = \sqrt{768/(225\pi) - 1}$,

$$\text{avg}(|E_{R,t}|) \left(1 - 1.960\sqrt{\frac{768-225\pi}{225\pi N}}\right) \leq \langle |E_{R,t}| \rangle \leq \text{avg}(|E_{R,t}|) \left(1 + 1.960\sqrt{\frac{768-225\pi}{225\pi N}}\right). \quad (147)$$

It is emphasized that these intervals are necessarily approximate, particularly for small N .

4.2.1.2 Standard Deviation To obtain confidence intervals for the sample standard deviation, we use the basic relation (31) and the expressions for the first (μ_X), second ($\mu_{2,X}$), and fourth ($\mu_{4,X}$) moments of the respective $\chi_{2p}^{(2)}$ parent pdfs [11]:

$$f_P(p) = \frac{p^{\frac{\nu}{2}-1} \exp\left(-\frac{p}{2\sigma^2}\right)}{2^{\nu/2} \sigma^\nu \Gamma(\nu/2)}, \quad (\nu = 2 \text{ or } 6) \quad (148)$$

$$f_{|E|}(|e|) = \frac{|e|^{\nu-1} \exp\left(-\frac{|e|^2}{2\sigma^2}\right)}{2^{\frac{\nu}{2}-1} \sigma^\nu \Gamma(\nu/2)}, \quad (\nu = 2 \text{ or } 6). \quad (149)$$

The moments of the $\chi_{2p}^{(2)}$ pdfs follow as

- for the Cartesian power $P_{R,\alpha}$:

$$\mu_{m,P_{R,\alpha}} = m! \mu_{P_{R,\alpha}}^m = m! \sigma_{P_{R,\alpha}}^m; \quad (150)$$

- for the vectorial power $P_{R,t}$:

$$\mu_{m,P_{R,t}} = \frac{(m+2)!}{2 \times 3^m} \mu_{P_{R,t}}^m = \frac{(m+2)!}{2 \times 3^{m/2}} \sigma_{P_{R,t}}^m; \quad (151)$$

- for the Cartesian field strength $|E_{R,\alpha}|$:

$$\mu_{m,|E_{R,\alpha}|} = \begin{cases} \frac{2^{2n}n!}{\pi^n} \mu_{|E_{R,\alpha}|}^{2n}, & m = 2n \\ \frac{2^{n-1}(2n+1)!!}{\pi^n} \mu_{|E_{R,\alpha}|}^{2n+1}, & m = 2n+1 \end{cases} \quad (152)$$

$$= \begin{cases} \frac{2^{2n}n!}{(4-\pi)^n} \sigma_{|E_{R,\alpha}|}^{2n}, & m = 2n \\ \frac{2^{n-1}(2n+1)!!}{(4-\pi)^n} \sqrt{\frac{\pi}{4-\pi}} \sigma_{|E_{R,\alpha}|}^{2n+1}, & m = 2n+1; \end{cases} \quad (153)$$

- for the vectorial field strength $|E_{R,t}|$:

$$\mu_{m,|E_{R,t}|} = \begin{cases} \frac{2^{8n-1}(n+2)!}{15^{2n}\pi^n} \mu_{|E_{R,t}|}^{2n}, & m = 2n \\ \frac{2^{7n}(2n+5)!!}{15^{2n+1}\pi^n} \mu_{|E_{R,t}|}^{2n+1}, & m = 2n + 1 \end{cases} \quad (154)$$

$$= \begin{cases} \frac{2^{8n-1}(n+2)!}{(768-225\pi)^n} \sigma_{|E_{R,t}|}^{2n}, & m = 2n \\ \frac{2^{7n}(2n+5)!!}{15 (768-225\pi)^n} \sigma_{|E_{R,t}|}^{2n+1}, & m = 2n + 1; \end{cases} \quad (155)$$

from which the central moments μ' can be calculated and with which we obtain the standard error for $\text{Std}(P_R)$ and $\text{Std}(|E_R|)$:

- for the Cartesian power $P_{R,\alpha}$:

$$s_{\text{Std}(P_{R,\alpha})} = \sqrt{\frac{2}{N-1}} \mu_{P_{R,\alpha}} = \sqrt{\frac{2}{N-1}} \sigma_{P_{R,\alpha}}; \quad (156)$$

- for the vectorial power $P_{R,t}$:

$$s_{\text{Std}(P_{R,t})} = \sqrt{\frac{1}{3(N-1)}} \mu_{P_{R,t}} = \sqrt{\frac{1}{N-1}} \sigma_{P_{R,t}}; \quad (157)$$

- for the Cartesian field strength $|E_{R,\alpha}|$:

$$s_{\text{Std}(|E_{R,\alpha}|)} = \sqrt{\frac{1}{\pi(N-1)}} \mu_{|E_{R,\alpha}|} = \sqrt{\frac{1}{(4-\pi)(N-1)}} \sigma_{|E_{R,\alpha}|}; \quad (158)$$

- for the vectorial field strength $|E_{R,t}|$:

$$s_{\text{Std}(|E_{R,t}|)} = \frac{8}{15} \sqrt{\frac{1}{\pi(N-1)}} \mu_{|E_{R,t}|} = \sqrt{\frac{64}{(768-225\pi)(N-1)}} \sigma_{|E_{R,t}|}. \quad (159)$$

The 95 % confidence intervals follow as:

- for the Cartesian power $P_{R,\alpha}$:

$$s_{P_{R,\alpha}} \left(1 - \frac{1.960 \sqrt{2}}{\sqrt{N-1}} \right) \leq \sigma_{P_{R,\alpha}} \leq s_{P_{R,\alpha}} \left(1 + \frac{1.960 \sqrt{2}}{\sqrt{N-1}} \right); \quad (160)$$

- for the vectorial power $P_{R,t}$:

$$s_{P_{R,t}} \left(1 - \frac{1.960}{\sqrt{N-1}} \right) \leq \sigma_{P_{R,t}} \leq s_{P_{R,t}} \left(1 + \frac{1.960}{\sqrt{N-1}} \right); \quad (161)$$

- for the Cartesian field strength $|E_{R,\alpha}|$:

$$s_{|E_{R,\alpha}|} \left(1 - \frac{1.960}{\sqrt{N-1}\sqrt{4-\pi}} \right) \leq \sigma_{|E_{R,\alpha}|} \leq s_{|E_{R,\alpha}|} \left(1 + \frac{1.960}{\sqrt{N-1}\sqrt{4-\pi}} \right); \quad (162)$$

- for the vectorial field strength $|E_{R,t}|$:

$$s_{|E_{R,t}|} \left(1 - \frac{1.960}{\sqrt{N-1}} \sqrt{\frac{64}{768-225\pi}} \right) \leq \sigma_{|E_{R,t}|} \leq s_{|E_{R,t}|} \left(1 + \frac{1.960}{\sqrt{N-1}} \sqrt{\frac{64}{768-225\pi}} \right). \quad (163)$$

4.2.2 Realistic Chamber ($N_{\max} \not\rightarrow +\infty$)

We now investigate the effect of the difference between the generated vs. maximum number of independent stir states on the confidence limits. For the average value of the transmitted (or received) power, relevant to emissions testing and chamber validation, typical relative levels of uncertainty for the estimate average total radiated power are of the order of $1/\sqrt{N}$: with (70), and $\sigma_{P_\alpha} = \langle P_\alpha \rangle$ for an idealized χ_2^2 distributed $P_{T,\alpha}$ or $P_{R,\alpha}$,

$$s_{\text{Avg}(P_\alpha)} = \frac{\text{avg}(P_\alpha)}{\sqrt{N}} \sqrt{\frac{N_{\max} - N}{N_{\max} - 1}} \quad (164)$$

compared to $s_{\text{Avg}(P_\alpha)} = \text{avg}(P_\alpha)/\sqrt{N}$. Instead of (144), a 95 % confidence interval for $\langle P_\alpha \rangle$ is now

$$\text{avg}(P_\alpha) \left(1 - \frac{1.960}{\sqrt{N}} \sqrt{\frac{N_{\max} - N}{N_{\max} - 1}} \right) \leq \langle P_\alpha \rangle \leq \text{avg}(P_\alpha) \left(1 + \frac{1.960}{\sqrt{N}} \sqrt{\frac{N_{\max} - N}{N_{\max} - 1}} \right) \quad (165)$$

i.e.,

$$\text{avg}(P_\alpha) \left(1 - \frac{1.960}{\sqrt{N}} \sqrt{1 - \frac{N - 1}{N_{\max} - 1}} \right) \leq \langle P_\alpha \rangle \leq \text{avg}(P_\alpha) \left(1 + \frac{1.960}{\sqrt{N}} \sqrt{1 - \frac{N - 1}{N_{\max} - 1}} \right). \quad (166)$$

If $N_{\max} \geq N \gg 1$, then this interval can be approximated as

$$\text{avg}(P_\alpha) \left(1 - 1.960 \sqrt{\frac{1}{N} - \frac{1}{N_{\max}}} \right) \leq \langle P_\alpha \rangle \leq \text{avg}(P_\alpha) \left(1 + 1.960 \sqrt{\frac{1}{N} - \frac{1}{N_{\max}}} \right) \quad (167)$$

where

$$N \leq N_{S_1} \cdot \dots \cdot N_{S_b} \cdot N_{L_T} \cdot N_{L_R} \cdot N_P \cdot N_F \cdot N_B \cdot \dots \triangleq \prod_{i=1}^n N_i, \quad (168)$$

$$N_{\max} \leq N_{\max,S_1} \cdot \dots \cdot N_{\max,S_b} \cdot N_{\max,L_T} \cdot N_{\max,L_R} \cdot N_{\max,P} \cdot N_{\max,F} \cdot N_{\max,B} \cdot \dots \triangleq \prod_{i=1}^n N_{\max,i} \quad (169)$$

are the actually generated (effective) number of independent stir states, N , and the theoretically maximum number of possible independent³⁸ stir states, $N_{\max}$, respectively, when the respective stir mechanisms are combined into a single *hybrid stirring process*, as proposed and analyzed in [3], viz., b mechanical stirrers with respective numbers of stir states $S_1, \dots, S_b$; $L_{T,R}$ independent (i.e., uncorrelated to any order) spatial locations of the transmitting or receiving antenna; P independent antenna polarizations (per antenna); F frequencies within the tolerable bandwidth around a specified center

³⁸Note that for hybrid, i.e., combined stir mechanisms, the independence must hold across all these mechanisms. For example, the combination of N_{L_T} and N_{L_R} may produce less than $N_{L_T} \cdot N_{L_R}$ overall independent states; two stirrers may produce less than $N_{S_1} \cdot N_{S_2}$ states, etc. For hybrid stirring, the value of N and $N_{\max}$ can be estimated, e.g., using the associated multi-dimensional correlation function for all stir mechanisms.

frequency, with frequency spacing sufficiently large to yield $N_{(\max),F}$ independent sample values; B bandwidths per frequency, etc.

Note that the mixing mechanisms considered in the definition of N and $N_{\max}$ do not need to show correspondence. For example, one may measure using just one single mechanical rotating stirrer and a single pair of locations for transmitting and receiving antennas ($N = N_S$), but use the resulting tuner sweep data to estimate the uncertainties for *arbitrary* locations of the antennas $N_{\max} \leq N_S \cdot N_{L_T} \cdot N_{L_R}$.

Provided the individual $N_{(\max),i}$ and sampled data are known, the values of N and $N_{\max}$ in (168)–(169) can be estimated, for example, from the Welch–Satterthwaite equation [32]–[34], based on calculated sample variances s_i^2 of the individual stir sweeps, as

$$N_{(\max)} \simeq \frac{(\sum_{i=1}^n c_i^2 s_i^2)^2}{\sum_{i=1}^n [(c_i^2 s_i^2)^2 / N_{(\max),i}]} \quad (170)$$

in which the sample standard deviations of the individual stirring processes can be given different weights (sensitivity coefficients) c_i , in case the different stirring mechanisms are not statistically equivalent, but which are here assumed to be equal, for simplicity ($c_i = 1/n$).

The question arises whether a finite upper limit for $N_{\max}$ exists, either theoretically or practically – i.e., whether the achievable uncertainty has a fundamental lower limit greater than zero for a given cavity, frequency, and (finite) bandwidth³⁹. In any case, $N_{\max}$ taken to be approaching infinity serves as a good first-order approximation for most practical applications and conditions of operation, even if only a single (say, the m th) stirring mechanism with a limited $N_{\max,m}$ is actually generated. For example, since the size of a spatial coherence cell is of the order of $(\lambda/2)^3$ (e.g., [40]), the number of cells within the working volume of the chamber, V , grows without bound for increasing frequency, whence in the limit for $N_{\max} \rightarrow +\infty$, (167) widens to

$$\text{avg}(P_T) \left(1 - \frac{1.960}{\sqrt{N}}\right) \leq \langle P_T \rangle \leq \text{avg}(P_T) \left(1 + \frac{1.960}{\sqrt{N}}\right). \quad (171)$$

Figure 9 shows the effect of a finite $N_{\max}$ on the location and width of a 95 % confidence interval for a Gauss normally distributed random variable, as an asymptotic approximation for the sample statistic for $\langle P_\alpha \rangle$. The narrowing of the confidence interval for decreasing $N_{\max}$ is apparent.

4.3 Effect of Impedance Mismatch of Emitting Source on Reflection Coefficient of Receiver

In chamber validation, both the source and detector are impedance matched when using a VNA. In this case, the S_{ii} correspond to the reflection coefficients Γ_i at the transmitter and receiver sides.

³⁹Note that although the mode density will be finite, for any stir state, there is in principle no limitation on the number of different eigenmodes (and, hence, local field values) that a composite stirring process can generate.

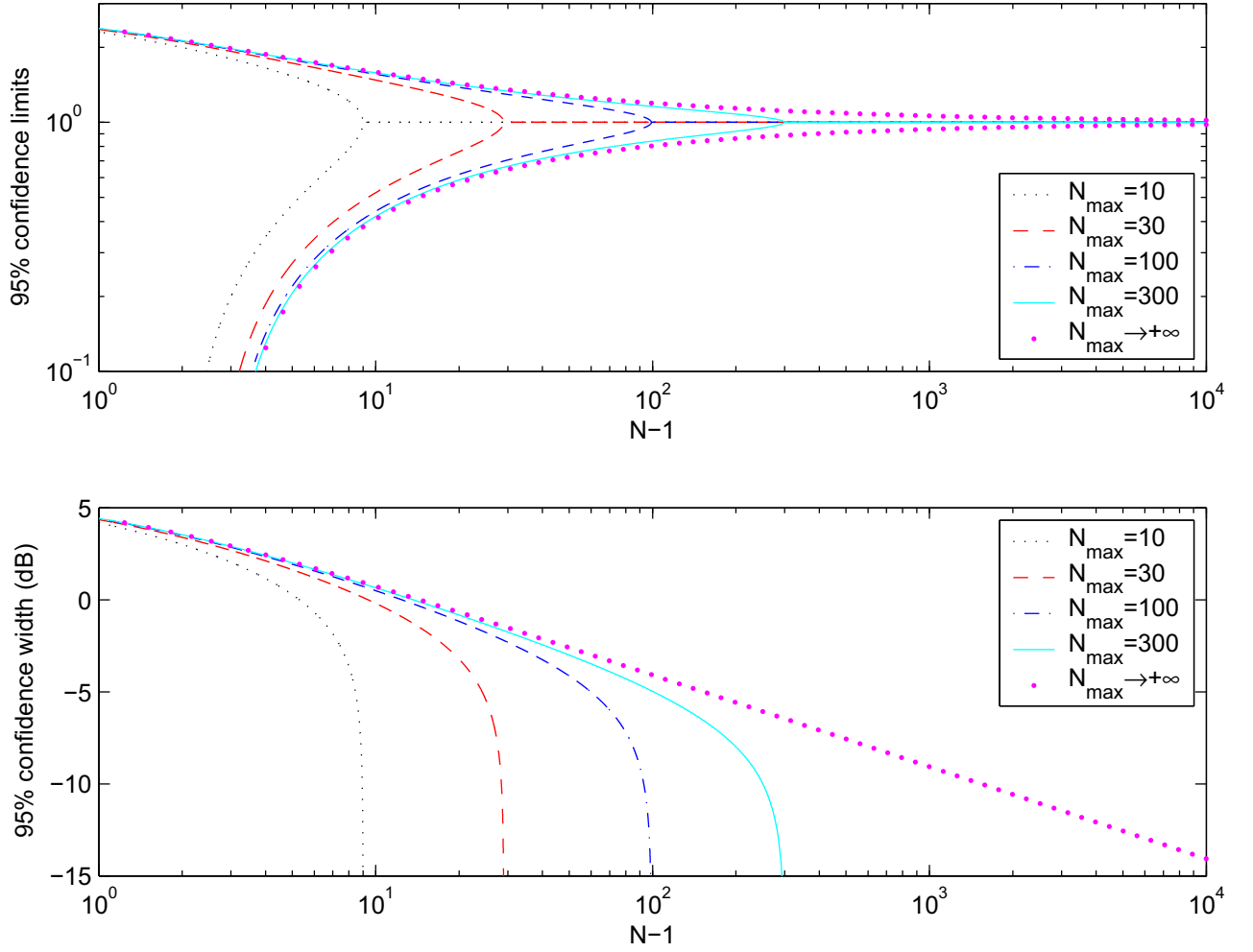


Figure 9: Effect of finite $N_{\max}$ on limits (top) and width (bottom) of a 95 % confidence interval for $\langle P_{\alpha} \rangle$.

During emissions measurements, however, typically only the receiving antenna is impedance matched to the EME and detector (and even so only on average), whereas the source will be significantly mismatched to the EME (at least for some frequencies within the emitted spectrum), particularly where it concerns an unintentional emitter. The average impedance mismatch is governed by the difference between the output impedance of the EUT, Z_S , and the input impedance of the chamber, Z_{ch} , at a chosen frequency. Owing to the on-average mismatch presented by the emitter, S_{22} can then no longer be identified with Γ_2 . Rather⁴⁰,

$$\Gamma_2 = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1 - S_{11}\Gamma_S} \quad (172)$$

and for the emitted (radiated) power generated by the internal source P_S of the emitter

$$P_T = (1 - |\Gamma_1|^2)P_S. \quad (173)$$

⁴⁰The analogy with an (overmoded) waveguide junction is valid, provided the aperture leakage and wall loss of the cavity can be neglected so that the system is closed (i.e., can be represented as a reservoir) with respect to EM energy.

The (energy) radiation efficiency of the emitting EUT, as measured by the chosen detector, is given by the power gain P_D/P_T [39]:

$$\epsilon_T = \frac{P_D}{P_T} = \frac{|S_{21}|^2(1 - |\Gamma_D|^2)}{(1 - |\Gamma_1|^2)|1 - S_{22}\Gamma_D|^2} = \frac{|S_{21}|^2(1 - |\Gamma_D|^2)}{|1 - S_{22}\Gamma_D|^2 - |(S_{12}S_{21} - S_{11}S_{22})\Gamma_D + S_{11}|^2}. \quad (174)$$

For the ideal case of an impedance matched receiver ($\Gamma_D = 0$), we retrieve

$$\epsilon_T = \frac{|S_{21}|^2}{1 - |S_{11}|^2}. \quad (175)$$

The amplitude of the net forward radiation from the EUT (considered as a source) into the cavity can be represented as

$$a_S^{\text{MT/MSRC}} = a_S^{\text{FS}} + \Gamma_1 \Gamma_S a_1 \quad (176)$$

where a_S^{FS} represents the net forward amplitude that would have been transmitted by the antenna matched to free space, i.e., for $\Gamma_1 = 0$, and Γ_S is the reflection coefficient of the source itself. Since an overmoded cavity is only on average impedance matched to a source, even for a transmitting antenna whose output impedance is matched to free space, we have in general that $\Gamma_1 \neq 0$ even if $\langle \Gamma_1 \rangle = 0$. This demonstrates why in general fluctuations of the injected field with respect to the stir state exist ($a_S^{\text{MT/MSRC}} \neq a_S^{\text{FS}}$).

4.4 Experimental Results

4.4.1 Uncertainties for Validation Measurements of a MTRC

Measurements were performed using a pair of logarithmic-periodic antennas inside the larger reverberation chamber (dimensions 6.55 m $\times$ 5.85 m $\times$ 3.50 m) at the U.K. National Physical Laboratory (NPL). Figure 10 shows sample statistics for chamber validation using experimental S-parameter data measured between 200 MHz and 1500 MHz. Except near the LUF, agreement with the values of the average and standard error for an ideal chamber is seen to be very good. The relatively small impedance mismatch factors ($1 - \langle |S_{ii,0}|^2 \rangle$) are seen to have a negligible effect on the uncertainty for $P_{T,0}$ in this case. The measured coefficient of variation of $\text{Avg}(P_T)$ is close to $1/\sqrt{N}$, in agreement with (144). Corresponding results for the range 1 – 18 GHz, obtained using a pair of dual ridge horn antennas, are shown in Fig. 11.

The measured cross-correlation function between $|S_{21}(f)|^2$ and $1 - |S_{11}(f)|^2$ is shown in Fig. 12. It can be seen that the measured are considerably smaller than the theoretical value +1, particularly at the higher frequencies.

The equivalent number of independent samples, N , is here defined and calculated from the first crossing of the 1/e-level by the autocorrelation function (ACF), with linear interpolation between

angular increments. Depending on whether the ACF for the real (or imaginary) component of S_{21} or for its magnitude or squared magnitude $|S_{21}|^{(2)}$ is used, the value of N thus obtained is slightly different (Fig. 13), particularly in the overlapping frequency band (1 – 1.5 GHz). This suggests that impedance mismatch of the antennas (in the bands 1.3 – 1.5 GHz and 1 – 2 GHz, respectively) and the antenna factors within these bands affect the ACF.

Figure 14 compares the deviation from unity of $\nu_T^2 - \nu_D^2$, as specified by (135). The increased sensitivity of S_{ii} to imperfections at higher frequencies is apparent. Nevertheless, an overall decrease of the deviation from unity is noticed, indicating a reduction of the level of fluctuations of $\text{Avg}(P_T)$, i.e., a progressively improving statistical impedance match, as expected. Also shown is the ratio of the actual $s_{\text{Avg}(P_T)}$ to the ideal $s_{\text{Avg}(P_T)}$. This ratio exhibits a steady decrease, indicating that the relative contribution of the impedance mismatch in the uncertainty of $\text{Avg}(P_T)$ becomes progressively smaller with increasing frequency.

The use of $|S_{21,0}|^2$ data (calculation assuming ideal chambers) only gives a deceptively smaller discrepancy, although the deviations are small in any case (less than 1%).

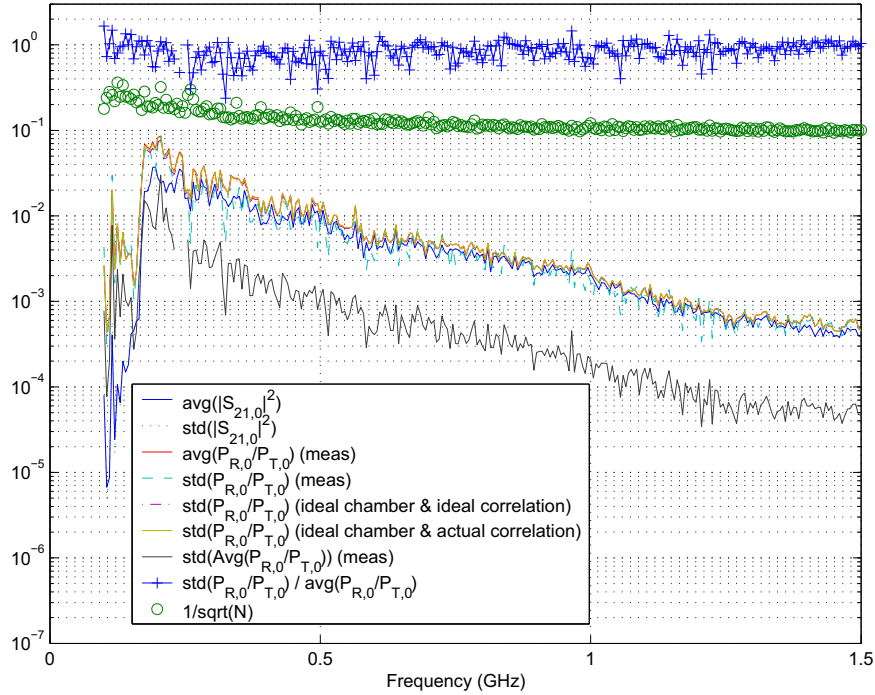


Figure 10: Measured sample statistics between 100 MHz and 1500 MHz. The curve marked “ideal chamber & ideal correlation” uses the measured $|S_{21}|^2$ data only with inference of $\rho_{|S_{21}|^2, 1-|S_{11}|^2} = 1$; the curve marked “ideal chamber & actual correlation” uses measured data for $|S_{11}|^2$, $|S_{22}|^2$ and $|S_{21}|^2$.

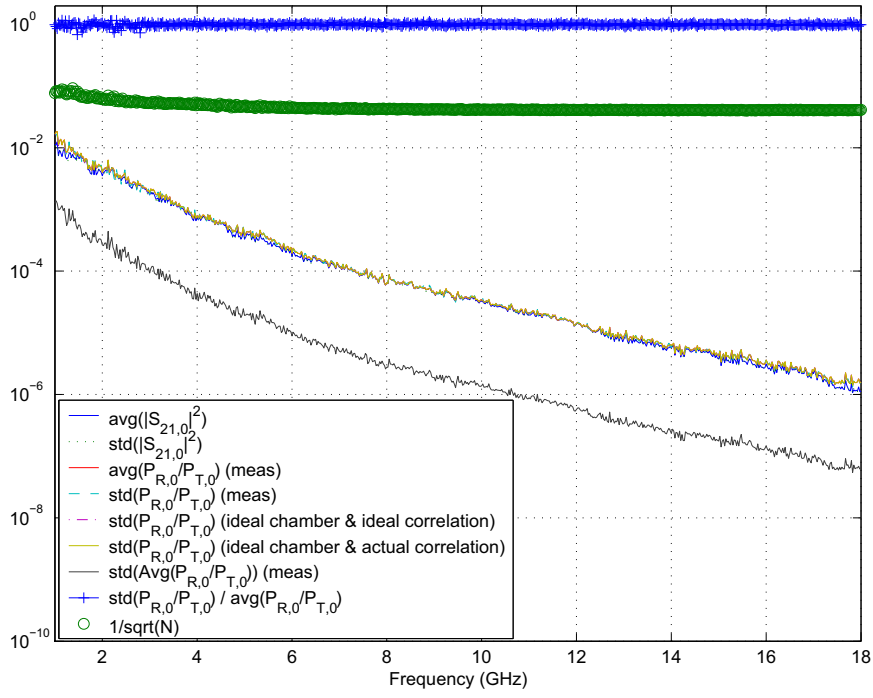
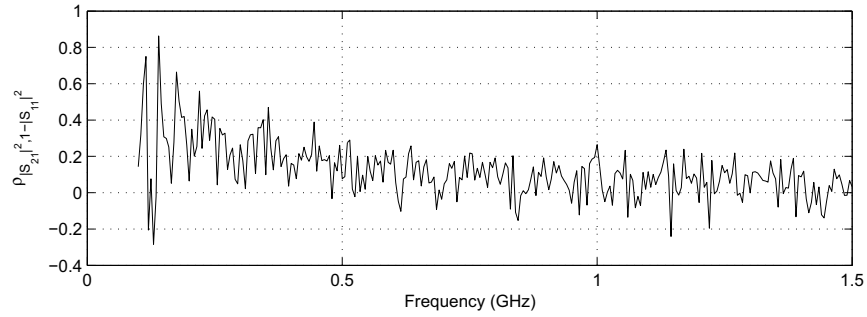
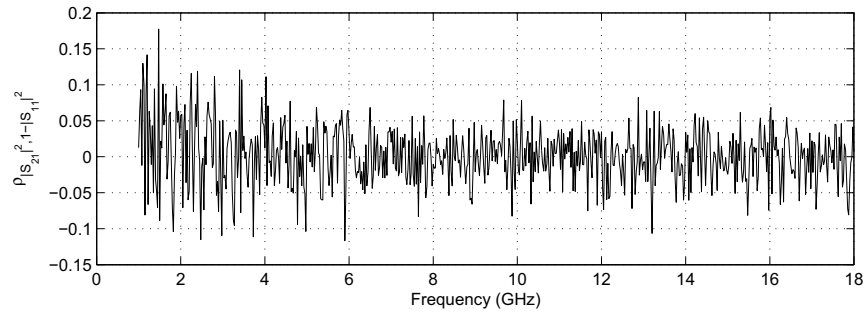


Figure 11: Measured sample statistics between 1 GHz and 18 GHz. The curve marked “ideal chamber & ideal correlation” uses measured $|S_{21}|^2$ data only with inference of $\rho_{|S_{21}|^2, 1-|S_{11}|^2} = 1$; the curve marked “ideal chamber & actual correlation” uses measured data for $|S_{11}|^2$, $|S_{22}|^2$ and $|S_{21}|^2$.



(a)



(b)

Figure 12: Measured cross-correlation function $\rho_{|S_{21}|^2, 1-|S_{11}|^2}(f)$: (a) between 100 and 1500 MHz; (b) between 1 and 18 GHz.

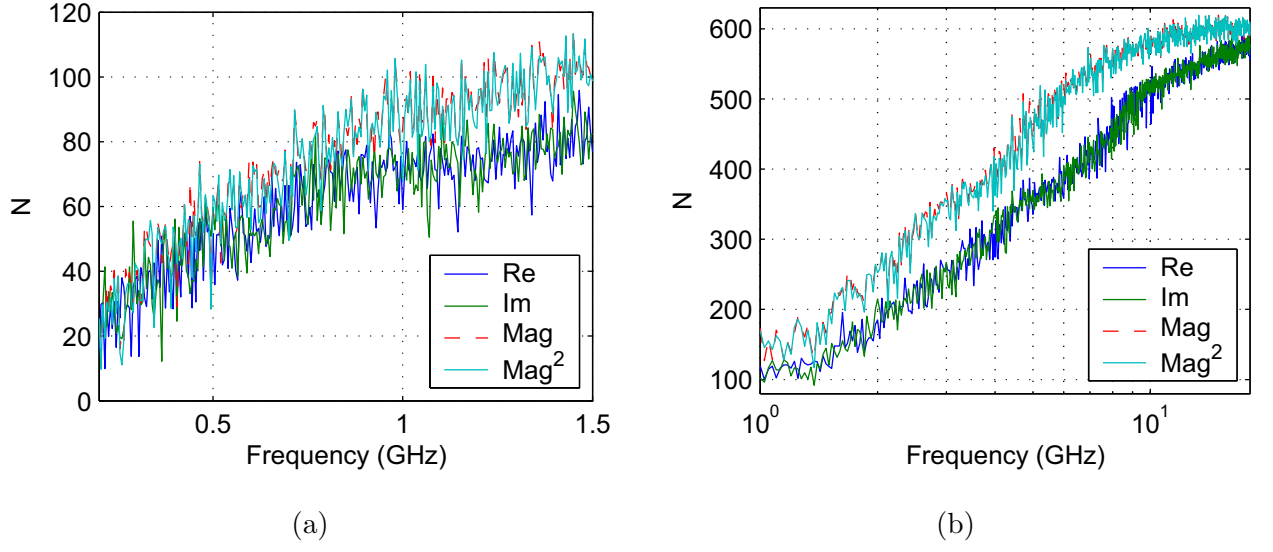


Figure 13: Estimated number of independent samples, based on $1/e$ -threshold crossing of the autocorrelation function, for $\Re(P_{R,0}/P_{T,0})$, $\Im(P_{R,0}/P_{T,0})$, $|P_{R,0}/P_{T,0}|$ or $|P_{R,0}/P_{T,0}|^2$: (a) from 200 to 1500 MHz, (b) from 1 GHz to 18 GHz.

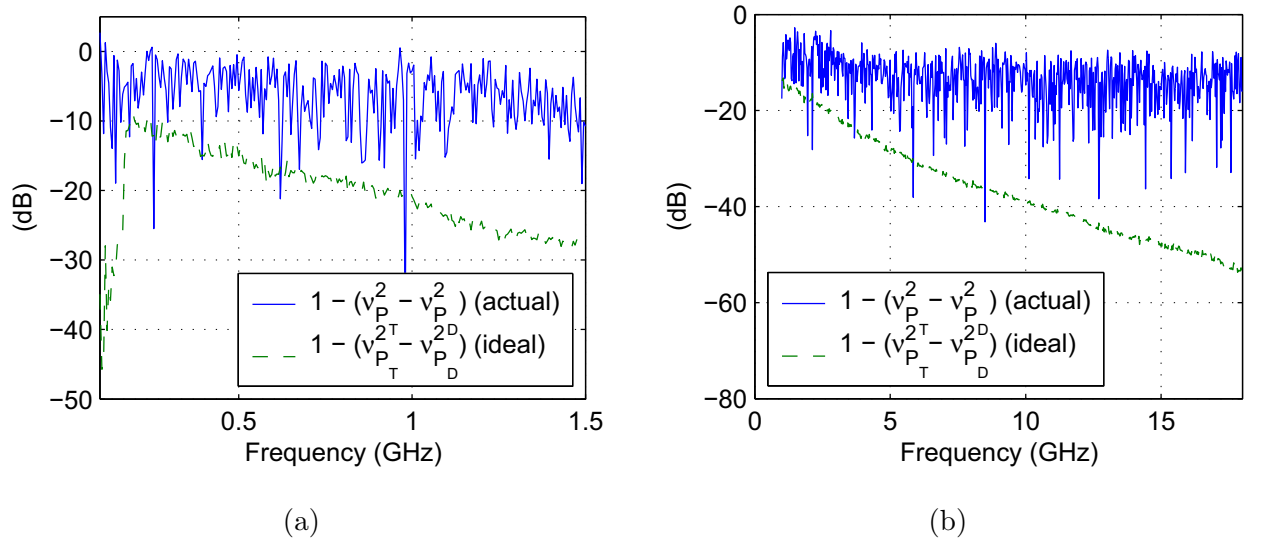


Figure 14: Deviation of $\nu_T^2 - \nu_D^2$ from unity for measured data, (a) for 100 MHz to 1.5 GHz, and (b) for 1 GHz to 18 GHz. The curves marked “ideal” use the measured $|S_{21}|^2$ data only; the curves marked “actual” use measured data for $|S_{11}|^2$, $|S_{22}|^2$ and $|S_{21}|^2$.

4.4.2 Application: Intrinsic Field Uncertainty for Emission Measurements

To apply the previous results, we used measured data for radiated emissions by a reference EUT (RefRad), operating in its “slot” radiating mode. This EUT produces EM radiation generated by an internal digital circuit (comb generator) and escaping through a rectangular slot in one of its side walls. The aperture is pointed away from the receiving antenna during the measurements. The characteristics and measured average radiated power of this artificial EUT have been reported previously [41] and showed good correlation with measured data previously obtained for the same EUT in other reverberation chambers, as part of a round-robin exercise [42, Fig. 6]. Here, the focus is on establishing confidence intervals for the measured data within the NPL chamber, to allow for a proper intercomparison between these data sets. Since the confidence interval is determined empirically, a proper intercomparison requires each laboratory to establish its own confidence interval.

Firstly, we consider the radiating EUT and receiving antenna at a single pair of locations, for 100 equiangular stirrer positions⁴¹ in frequency steps of 5 MHz from 100 MHz to 1300 MHz. The receiving antenna is a linearly polarized log-periodic antenna, thus capturing only one Cartesian component of the electric field. In order to estimate the vectorial transmitted electric power from this Cartesian power, the estimated $\text{avg}(P_T)$ should be multiplied by three (4.77 dB to be added). Figure 15(a) shows the estimated average radiated power (Cartesian component) $\text{avg}(P_T)$ for a single arbitrary location of the EUT, together with 95% confidence intervals estimated as $\text{avg}(P_T)(1 \pm 1.96n_{P_T})$. The value of n_{P_T} as an estimate for ν_{P_T} is calculated either from (135) (curves marked “ideal”); from (135) but with $[1/(1 - 2\langle|S_{21,0}|^2\rangle)] - 1$ replaced by $\sigma_{1-|S_{ii,0}|^2}/(1 - \langle|S_{ii,0}|^2\rangle)$ (curves marked “actual”); and from (136) but also with $[1/(1 - 2\langle|S_{21,0}|^2\rangle)] - 1$ replaced by $\sigma_{1-|S_{ii,0}|^2}/(1 - \langle|S_{ii,0}|^2\rangle)$ (curves marked “actual & no correlation”). The averages and standard deviations are estimated from the measured values for $|S_{11,0}|^2$, $|S_{22,0}|^2$, $|S_{21,0}|^2$, $|S_{22}|^2$, and P_D . It can be seen that, except for very low frequencies, the confidence intervals obtained with all three methods are similar and gradually narrowing with increasing frequency.

Secondly, the EUT was placed and measured in two more positions (i.e., three spatial locations in total, $M = 3$), allowing for spatial averaging of the data to reduce MU. Figure 15(b) compares the average values and confidence intervals between a single location (black curves) and the spatial average (red curves). The standard deviation for P_T was here calculated according to the “actual & no correlation” method. The variation of $\text{avg}(P_{T,\alpha})$ with frequency is almost unchanged, while the

⁴¹Not all these positions are statistically independent at all measurement frequencies. The ACF data for the validated chamber was employed to establish the applicable value of N at each frequency.

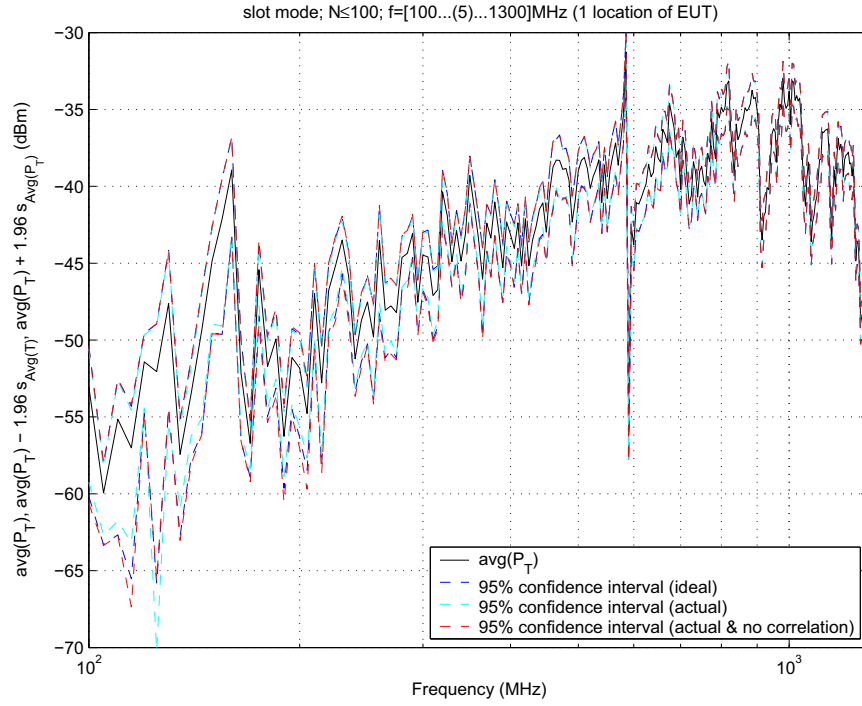
confidence interval narrows⁴² because of the triple spatial averaging. The effect is particularly visible near the resonance peak around 590 MHz. Additional measurements at a significantly larger number of spatial locations would be needed to make the interval still narrower.

The relative uncertainties ν_{P_T} and $\nu_{\text{Avg}(P_T)}$, again based on the measured values of ν_{P_D} , N , $|S_{21,0}|^2$, $|S_{11,0}|^2$, $|S_{22,0}|^2$ and $|S_{22}|^2$, are shown in more detail in Fig. 16 as a function of frequency, for a single location of the EUT ($M = 1$) and based on 100 generated stir states (i.e., $N(f) \leq 100$). This Figure shows the decrease of the relative uncertainty with increasing frequency, with asymptotic values and frequency dependence of ν_{P_T} being reached from around 250 MHz onwards. The absence of correlation is seen to give rise to a slower approach to asymptotic value for ν_{P_T} .

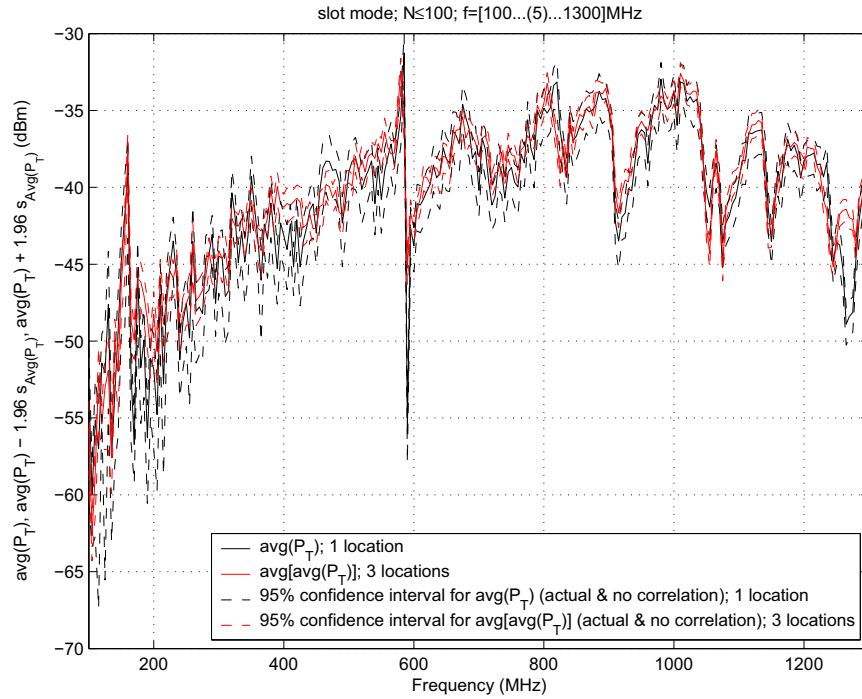
Regarding the estimated maximum value of the received emitted power (maximum with respect to all stir states, at a single location), the measured results in Fig. 17(a) show that the maximum-to-average ratio is fairly stable around 7.15 dB, as expected for a well-stirred chamber with $N = 100$ (cf. (183), (187), and (244), below). Subtle differences in the frequency characteristic of the measured maximum emitted power, compared to the average emitted power, can be noticed from detailed inspection of Fig. 17(b). In particular, the maximum value for the radiated power is difficult to estimate accurately, because at any single stir state (in particular the state at which the sample maximum value is measured) the plane-wave conditions and free-space value of the wave impedance do not hold instantaneously, i.e., $\max(P_R) \neq \max(|E_R|^2)/\eta_0$ even though $\langle \text{Max}(P_R) \rangle = \langle \text{Max}(|E_R|^2) \rangle / \eta_0$. Even if $\eta(\ell^*)$ can be determined for the particular stir state ℓ^* at which the power received by the monitoring antenna or field probe reaches a maximum, this is unlikely to be the same state (and, hence, the same value of η) at which the field at any other location will be maximal.

Corresponding results for the estimated average and maximum field strength for the emitted radiation are shown in Fig. 18. The maximum-to-average ratio is now centered around 8.20 dB, again in good agreement with theoretical value (cf. (184), (189), and (240), below).

⁴²by a factor $\sqrt{3}$ (~ 2.4 dB). Note that the black intervals should not (and do not, in general) necessarily straddle the red intervals, because the black confidence interval applies to (stir) averaging at a *local* field, i.e., when considered as a function of *both* stirred and spatial states, this average local power has $MN-1 = 0$ degrees of freedom, producing an infinitely wide interval, whereas the stirred and spatially averaged received power has $3N-1$ degrees of freedom.



(a)



(b)

Figure 15: Estimated average power radiated by EUT RefRad in slot mode, together with estimated 95% confidence intervals for (a) a single location of the EUT, and (b) for one location or averaging across three measurement locations of the EUT. The curves marked “ideal” assume (97) and only use measured values of $|S_{21,0}|^2$ with correlation coefficients for chamber validation equalling +1; the curves marked “actual” use all four measured S-parameter values with actual correlation coefficients.

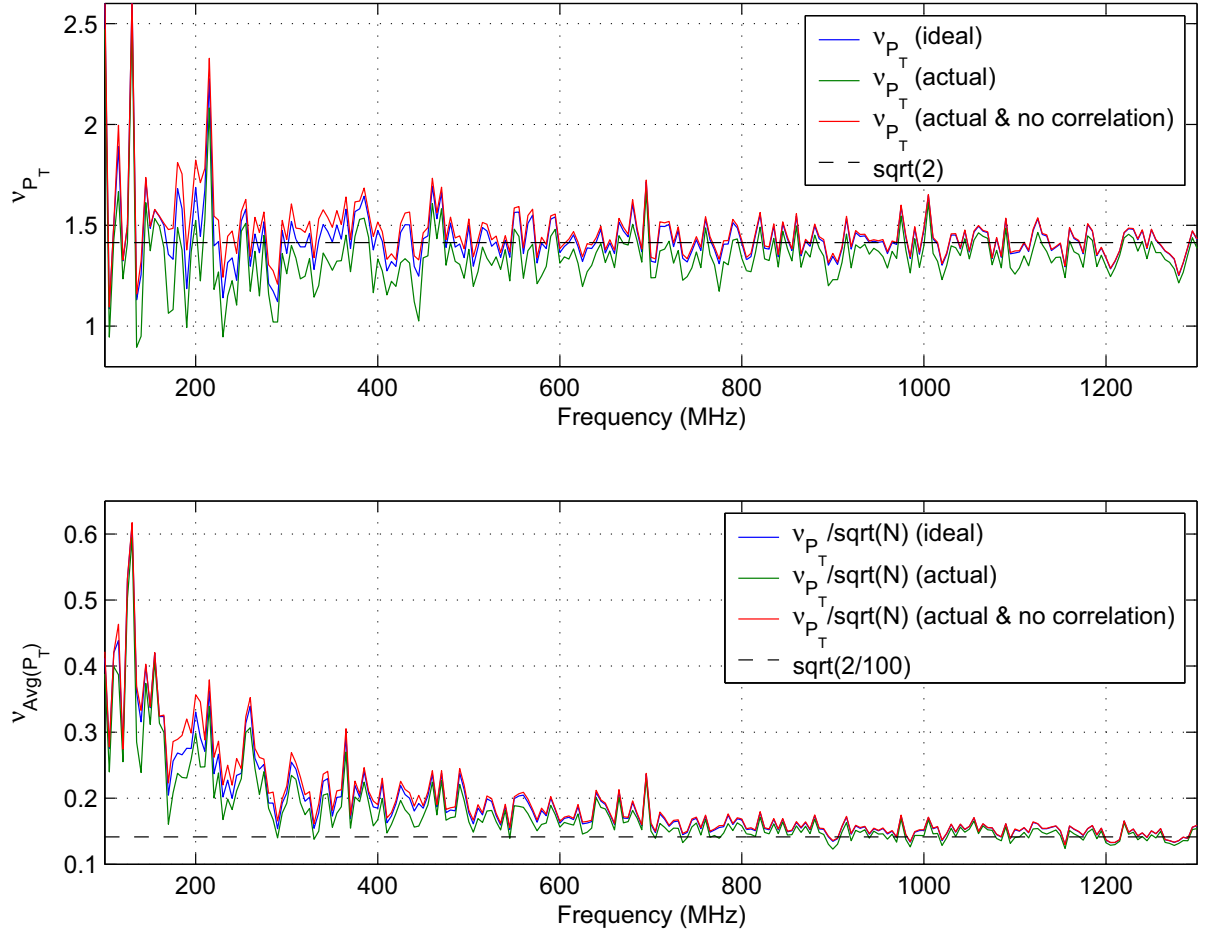
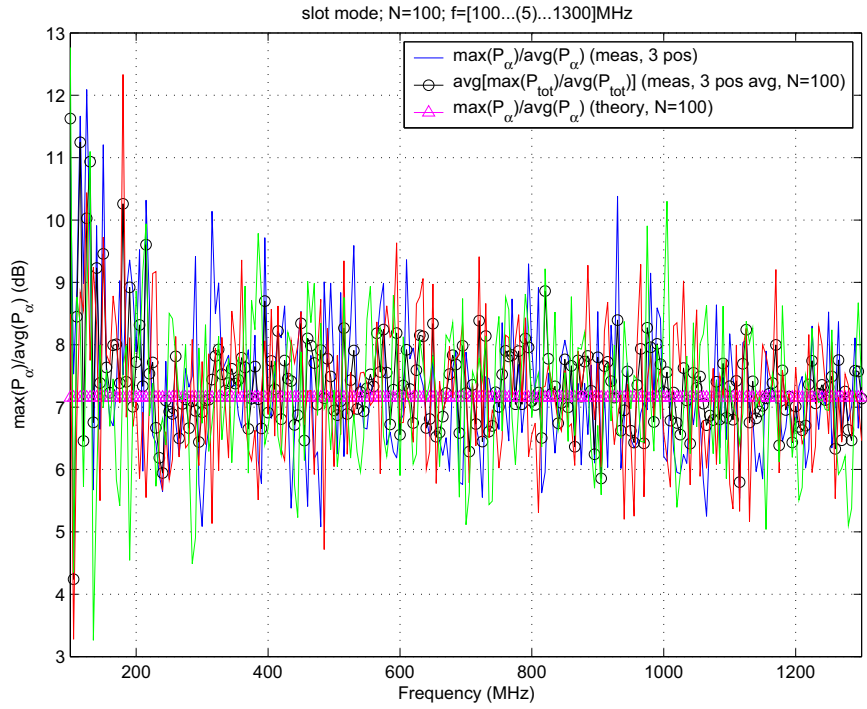
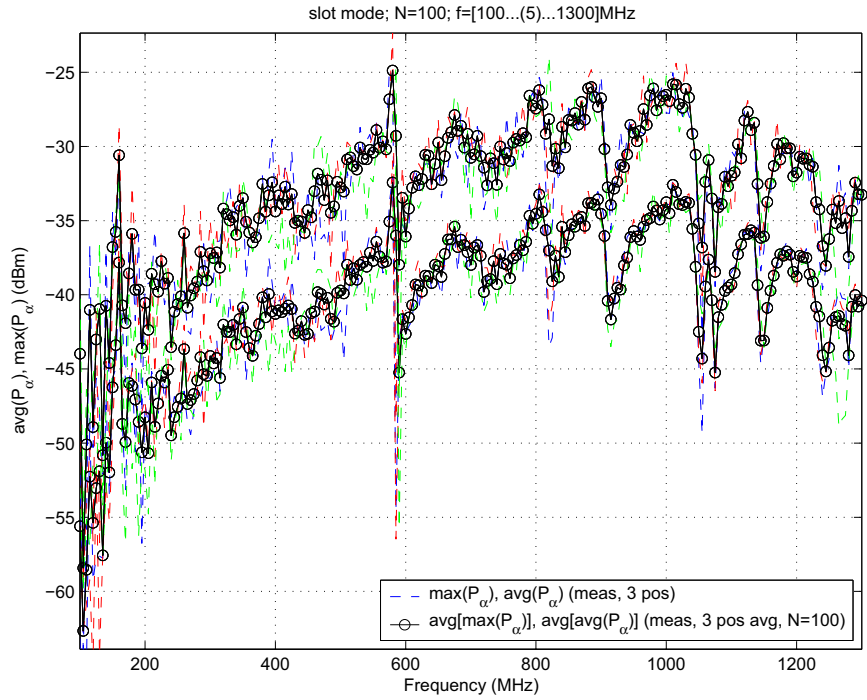


Figure 16: Dependence of ν_{P_T} and $\nu_{Avg(P_T)}$ on frequency, compared to respective theoretical values $\sqrt{2} \simeq 1.4142$ and $\sqrt{2}/\sqrt{N} \simeq 0.141$, for an emitting EUT at a single location. For evaluations involving M multiple locations of the EUT, the values are to be further divided by $\sqrt{M}$.

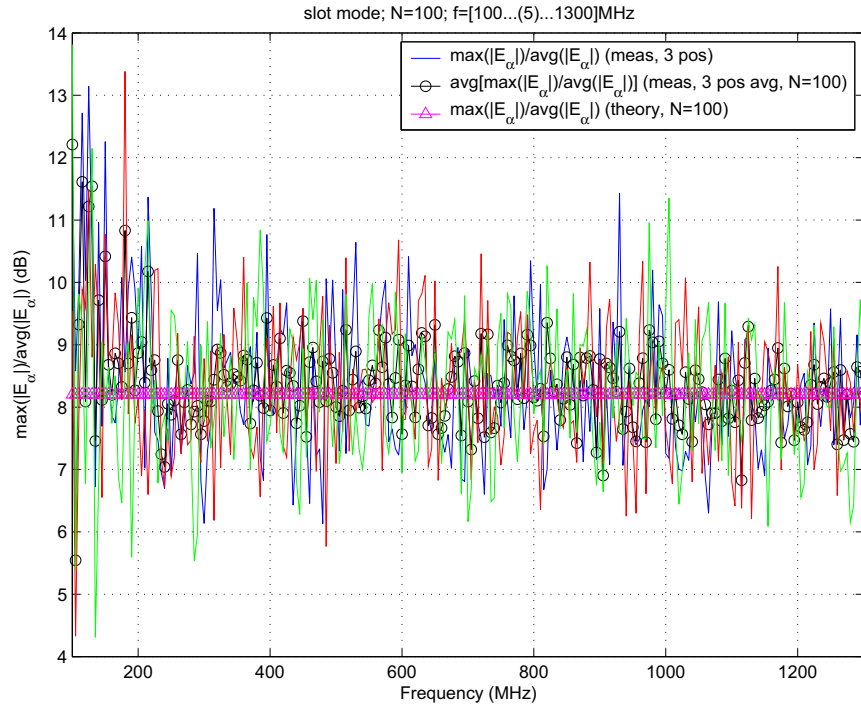


(a)

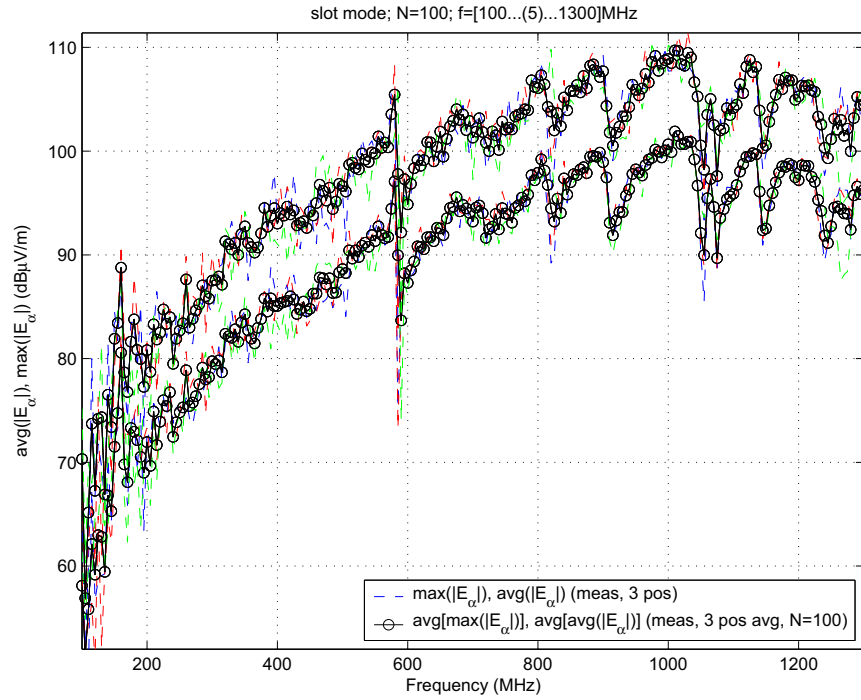


(b)

Figure 17: Maximum, average and maximum-to-average ratio of received power. In Fig. (b), the upper four curves show the characteristics max(P_{tot}) for the three locations of the EUT at a single receiver location (colored dashed lines) together with their 3-point spatial average per frequency (black line and circles); the lower four curves show the corresponding characteristics for avg(P_{tot}).



(a)



(b))

Figure 18: Maximum, average and maximum-to-average ratio of received electric field strength (Cartesian component). In Fig. (b), the upper four curves show the characteristics $\max(|E_\alpha|)$ for the three locations of the EUT at a single receiver location (colored dashed lines) together with their 3-point spatial average per frequency (black line and circles); the lower four curves show the corresponding characteristics for $\text{avg}(|E_\alpha|)$.

5 Immunity

5.1 General Considerations

Generally, in immunity measurements, the criterion for pass/failure determines whether the EUT shows a sign of failure when illuminated by a known field strength or power. The evaluation of uncertainty levels as given below does not address the uncertainty in the process of interpreting the sign of such failure, but only addresses the uncertainty in setting up the given power or field.

Immunity testing in MTRCs is usually performed with reference to the maximum field strength or the maximum energy or power density at a particular stir state during one rotation of the mode tuner. These maximum values are themselves random (statistical) functions of spatial position or sampling. Unlike in other EMC test methods, their randomness mainly originates from the randomness of the mode-stirred reverberant field itself. Therefore, the associated uncertainties are considerably larger than in the case of emissions. We shall assume that the spatial susceptibility of the EUT is 1-dimensional, i.e., of the wire type whence only a Cartesian (rectangular) component of the vector field is of relevance. Extensions to 2- or 3-dimensional test fields (e.g., relevant to testing of circuitry on printed circuit boards (PCBs) or thermally induced malfunction) are possible.

In connection with the experimental determination of the maximum value via mode tuning, some important facts concerning immunity testing should be noted. Firstly, during the test, the maximum value is measured by a monitoring sensor, placed inside the chamber and sufficiently far away from the EUT in order to avoid mutual coupling. At the location of the sensor, this value will be reached at some *different* stir state (angular position of the paddle wheel) than the state at which the EUT at a different location experiences its maximum test field, because the mode-tuned field is a random vector function of both stir state and spatial location. Nevertheless, across one full rotation, the maximum value should be the same (within the appropriate confidence interval for the maximum field whose width depends on N) at *all* locations inside the chamber, if strict-sense ergodicity holds. Consequently, the particular tuner state at which the test field will reach a maximum *at the location of the EUT* is in essence unpredictable. Therefore, the dwell time at each angle of the paddle wheel needs to be identical, in order to ensure adequate exposure to the maximum value at its unknown instance of occurrence. Any tuning of the paddle wheel position to find the maximum test level is only meaningful for obtaining a more accurate empirical sample value of the maximum test level for a sufficiently large N , i.e., it can only be indicative.

Secondly, and related to the previous remark, an immunity test inside a MT/MSRC exposes at any one time just one particular *point* location or (part of a) circuit component to the maximum field.

Within the EUT, this point location of maximum exposure therefore moves through space during the stirring process, in a more or less unpredictable manner, while the field strength at other locations within the EUT is more or less uncorrelated to this value at that instance. This is contrary to an immunity test inside a plane- or spherical-wave EME (AC, OATS, etc.), where all points located on a wavefront experience the same maximum field strength, with the field strength at points in front or behind this front related via the well-known $1/r$ dependence. In this sense, the MT/MSRC test is less⁴³ thorough, because superposition of high field strengths or induced currents at multiple locations can add up to produce a more detrimental effect. The detailed analysis of this effect requires consideration of the averaging effects, relative to the spatial or temporal scale of the test object and the wavelength or period of the excitation [22]. Clearly, the discrepancy increases for smaller wavelengths.

For an unspecified EUT in immunity testing, the pertinent quantity is the available power gain which defines the (energy) reception efficiency as

$$\epsilon_R = \frac{P_R}{P_S} = \frac{|S_{12}|^2(1 - |\Gamma_S|^2)}{|1 - S_{11}\Gamma_S|^2(1 - |\Gamma_2|^2)}. \quad (177)$$

Typically, the transmitting antenna (but not the irradiated EUT) is impedance matched ($\Gamma_S = 0$), in which case

$$\epsilon_R = \frac{P_R}{P_S} = \frac{|S_{12}|^2}{1 - |S_{22}|^2}. \quad (178)$$

With regard to the *field*, the transfer function between the source and EUT is

$$\varphi_R = \frac{E_R}{E_T} = \frac{E_R^+ + E_R^-}{E_T^+ + E_T^-} = \frac{S_{12}(1 + \Gamma_{EUT})}{(1 - S_{22}\Gamma_{EUT}) + S_{11}(1 - S_{22}\Gamma_{EUT}) + S_{12}S_{21}\Gamma_{EUT}}. \quad (179)$$

For an impedance matched EUT or detector ($\Gamma_{EUT} = 0$), $E_R/E_T = S_{12}/(1 + S_{11})$.

5.2 Ensemble Average, Median, Standard Deviation, Coefficient of Variation, Maximum Value, and Confidence Interval for Power or Field Strength

Theoretical and experimental results for the statistics of the maximum power and field strength have previously been presented elsewhere; cf., e.g., [15], [16], [35] for background. Here, the emphasis is on expressing the ensemble statistics of maximum values exclusively in terms of measurement-based physical parameters of the EME. This assists with the practical evaluation of ensemble statistics during the estimation of corresponding sample statistics from measured data.

First, we calculate the statistics for the ensemble field and power of the parent distribution. Given a known reference transmitted (i.e., net forward) input power P_T (or, more precisely, its average value

⁴³However, from the point of view of generating different angles of incidence and field polarization, the test is more thorough and exhibits greater spatial uniformity (in the statistical sense).

$\langle P_T \rangle$ over one period of the stirring process), then⁴⁴

$$\langle P_R \rangle = \frac{\eta_{T,0} Q A_e}{k V} \langle P_T \rangle = \frac{\eta_{T,0} \lambda^3 Q}{16\pi^2 V} \langle P_T \rangle. \quad (180)$$

For testing under idealized conditions of chamber operation – whereby the EM absorption or leakage via the antenna(s), apertures, cables, and EUT can be neglected compared to the ohmic absorption by the cavity walls – $\eta_{T,0} = 1$ and the prefactor of $\langle P_T \rangle$ in (180) can be *a priori* calculated with the aid of the asymptotic result [36] (cf. Sec. 4.1.1 for definition of symbols)

$$Q(\lambda) = \frac{3 \mu_0 V}{2 \mu_w S \delta_w} \quad (181)$$

yielding

$$\frac{\langle P_T \rangle}{\langle P_R \rangle} = \frac{16\pi^2 V}{\lambda^3 Q(\lambda)} = \frac{32 S}{3 c^3 \mu_0} \sqrt{\frac{\pi^3 \mu_w f^5}{\sigma_w}}. \quad (182)$$

For an ideal χ_2^2 distributed Cartesian $P_{R,\alpha}$ corresponding to an unbiased circular Gauss normal electric field $E_{R,\alpha}$, we have that $\langle P_{R,\alpha}^2 \rangle = 2 \langle P_{R,\alpha} \rangle^2$, whence

$$\sigma_{P_{R,\alpha}} = \langle P_{R,\alpha} \rangle = \frac{\lambda^3 \eta_{T,0} Q}{16\pi^2 V} \langle P_T \rangle. \quad (183)$$

With $\langle P_{R,t} \rangle = 3 \langle P_{R,\alpha} \rangle = 3 \langle |E_{R,\alpha}|^2 \rangle \lambda^2 / (8\pi \eta_0)$ and $\langle |E_{R,\alpha}|^2 \rangle = (4/\pi) \langle |E_{R,\alpha}| \rangle^2$, based on an idealized spatial-statistically homogeneous and isotropic (uniform) energy density throughout the stirred chamber, the corresponding ensemble mean and standard deviation of the average Cartesian field strength $|E_{R,\alpha}|$ are

$$\langle |E_{R,\alpha}| \rangle = \sqrt{\frac{\lambda \eta_0 \eta_{T,0} Q}{24 V} \langle P_T \rangle}, \quad (184)$$

$$\sigma_{|E_{R,\alpha}|} = \sqrt{\frac{(4 - \pi) \lambda \eta_0 \eta_{T,0} Q}{24\pi V} \langle P_T \rangle}. \quad (185)$$

with $\eta_0 \simeq 120\pi \simeq 377 \Omega$. With $\langle |E_{R,\alpha}|^2 \rangle = 2 \sigma_{E'_{R,\alpha}}^2$, the standard deviation for the in-phase or quadrature component of the unbiased circular Gauss normal complex field is [38]

$$\sigma_{E'_{R,\alpha}} = \sqrt{\frac{\lambda \eta_0 \eta_{T,0} Q}{12\pi V} \langle P_T \rangle}. \quad (186)$$

From this fundamental result, all the statistics of the maximum field strength and intensity follow.

⁴⁴Here, it is assumed that a monitoring receiving antenna or field probe is either (i) not present, (ii) maximally efficient ($\eta_R = 1$, i.e., lossless), or (iii) identical to the antenna or probe used during the chamber validation ($\eta_R = \eta_{R,0}$). If none of these conditions are satisfied, the results in this subsection differ by a constant factor that depends on the ratio $\eta_{R,0}/\eta_R$. In immunity testing of an EUT, η_R is incorporated in P_R .

The expected value⁴⁵ and standard deviation of the ensemble Cartesian maximum received power (i.e., for an infinite number of sample sets each consisting of N independent stir states) are [15], [16]

$$\langle \text{Max}(P_{R,\alpha}) \rangle = \frac{\lambda^3 \eta_{T,0} Q}{16\pi^2 V} \langle P_T \rangle \sum_{i=1}^N i^{-1} \simeq \frac{\lambda^3 \eta_{T,0} Q}{16\pi^2 V} \langle P_T \rangle \left[0.5772 + \ln(N+1) - \frac{1}{2(N+1)} \right] \quad (187)$$

$$\sigma_{\text{Max}(P_{R,\alpha})} = \frac{\lambda^3 \eta_{T,0} Q}{16\pi^2 V} \langle P_T \rangle \sqrt{\sum_{i=1}^N i^{-2}} \simeq \frac{\lambda^3 \eta_{T,0} Q}{16\pi^2 V} \langle P_T \rangle \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}. \quad (188)$$

For the maximum field strength, the estimate of its sample average

$$\langle \text{Max}(|E_{R,\alpha}|) \rangle \simeq \sqrt{\langle \text{Max}(|E_{R,\alpha}|^2) \rangle} \quad (189)$$

is a reasonable approximation for $\chi_2^{(2)}$ pdfs⁴⁶. Hence, with $\langle \text{Max}(|E_{R,\alpha}|^2) \rangle = 8\pi\eta_0 \langle \text{Max}(P_{R,\alpha}) \rangle / (3\lambda^2)$,

$$\langle \text{Max}(|E_{R,\alpha}|) \rangle \simeq \sqrt{\frac{\lambda \eta_0 \eta_{T,0} Q}{6\pi V} \langle P_T \rangle \left[0.5772 + \ln(N+1) - \frac{1}{2(N+1)} \right]}. \quad (190)$$

Thus, when $N \rightarrow +\infty$,

$$\langle \text{Max}(|E_{R,\alpha}|) \rangle \propto \sqrt{\ln(N)} \quad (191)$$

which increases more slowly than $1/\sqrt{N}$ as N increases. For the corresponding standard deviation, from (187)–(188),

$$\nu_{\text{Max}(P_{R,\alpha})} \triangleq \frac{\sigma_{\text{Max}(P_{R,\alpha})}}{\langle \text{Max}(P_{R,\alpha}) \rangle} \simeq \frac{\sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}}{0.5772 + \ln(N+1) - \frac{1}{2(N+1)}} \quad (192)$$

and using the approximation⁴⁷

$$\nu_{\text{Max}(|E_{R,\alpha}|)} \simeq \frac{\nu_{\text{Max}(P_{R,\alpha})}}{2} \quad (193)$$

together with the approximation (189), we obtain

$$\sigma_{\text{Max}(|E_{R,\alpha}|)} \simeq \sqrt{\frac{\lambda \eta_0 \eta_{T,0} Q}{24\pi V} \frac{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}{0.5772 + \ln(N+1) - \frac{1}{2(N+1)}} \langle P_T \rangle} \quad (194)$$

⁴⁵The approximation in (187), as obtained by truncating the Euler–Maclaurin formula for the finite series after the third term, is accurate to 2% at $N = 1$, with increasing accuracy for increasing N [16]. For noninteger values of N (equivalent number of independent stir states), the expected value and standard deviation can be calculated exactly by considering the $\chi_2^{(2)}$ distribution as a special case of the gamma distribution.

⁴⁶For a $\chi_2^{(2)}$ pdf, this approximation is accurate to within 2%, 1%, and 0.5% of the exact values for $N = 10$, 50, and 400, respectively.

⁴⁷For $\chi_2^{(2)}$ distributions, this approximation is accurate to within 2.4%, 3.2%, 3%, and 2.6% of the exact value for $N = 10$, 100, 1000, and 10000, respectively.

i.e.,

$$\sigma_{\text{Max}(|E_{R,\alpha}|)} \propto \frac{1}{\sqrt{\ln(N)}} \quad (195)$$

when $N \rightarrow +\infty$, rather than $\propto 1/\sqrt{N}$ as in the case of $\sigma_{\text{Avg}(|E_{R,\alpha}|)}$. This results in comparatively larger and more slowly diminishing uncertainty for the maximum field, compared to the uncertainty for the average field.

Although the asymptotic dependencies of the average or standard deviation of $\text{Max}(|E_{R,\alpha}|)$ on N are different from those for $\text{Max}(P_{R,\alpha})$, the relative uncertainty of both these test levels, as expressed by the coefficient of variation, has a similar qualitative asymptotic dependence⁴⁸: from (190), (193), and (194), we find that

$$\nu_{\text{Max}(P_{R,\alpha})} \simeq 2 \nu_{\text{Max}(|E_{R,\alpha}|)} \rightarrow \frac{\pi}{\sqrt{6} [0.5772 + \ln(N)]} \quad (196)$$

when $N \rightarrow +\infty$. Some relevant functional dependencies on N are qualitative compared in Fig. 19(a).

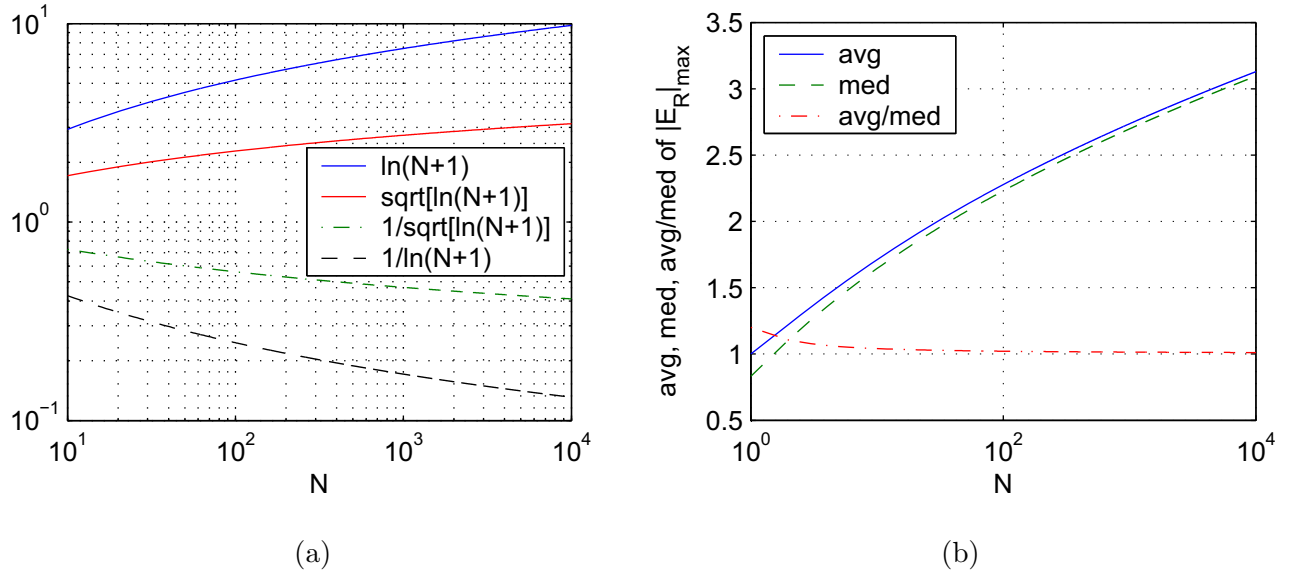


Figure 19: (a) Functions of N relevant to the expression of measures for centrality and uncertainty in immunity testing. (b) Average (avg) and median (med) of $\text{Max}(|E_{R,\alpha}|)$ together with their ratio as a function of the number of statistically independent stir states N .

An $\eta\%$ confidence interval for $\text{Max}(|E_{R,\alpha}|)$ has boundaries given by

$$\xi_{(1 \pm \eta/100)/2}^{\pm} \simeq \sqrt{\frac{\lambda \eta_0 \eta_{T,0} Q}{6\pi V} \langle P_T \rangle \ln \left\{ 1 - \left[\frac{1}{2} \left(1 \pm \frac{\eta}{100} \right) \right]^{\frac{1}{N}} \right\}^{-1}} \quad (197)$$

⁴⁸For practical use, the term 0.5772 is explicitly retained in (196) because of the slow increase of $\ln(N)$ towards infinity.

where we have used the fact that $\xi_{(1\pm\eta/100)/2}^{\pm}(\text{Max}(|E_{R,\alpha}|)) = \sqrt{\xi_{(1\pm\eta/100)/2}^{\pm}(\text{Max}(|E_{R,\alpha}|^2))}$ for $\chi_2^{(2)}$ distributions. This confidence interval is centered with respect to the median value of $\text{Max}(|E_{R,\alpha}|)$. The median is the 50 % percentile, i.e.,

$$\xi_{0.5} \simeq \sqrt{\frac{\lambda \eta_0 \eta_{T,0} Q}{6\pi V} \langle P_T \rangle \ln \left[1 - \left(\frac{1}{2} \right)^{\frac{1}{N}} \right]^{-1}}. \quad (198)$$

For $N \gg 1$, the median (198) is approximately equal to, but marginally smaller than $\langle \text{Max}(|E_R|) \rangle$ [Fig. 19(b)] and has a larger standard error [12]. The width of the confidence interval can be expressed either as the ratio of the upper to lower percentiles,

$$\frac{\xi_{(1+\eta/100)/2}^+}{\xi_{(1-\eta/100)/2}^-} = \sqrt{\frac{\ln \left\{ 1 - \left[\frac{1}{2} \left(1 + \frac{\eta}{100} \right) \right]^{\frac{1}{N}} \right\}}{\ln \left\{ 1 - \left[\frac{1}{2} \left(1 - \frac{\eta}{100} \right) \right]^{\frac{1}{N}} \right\}}} \quad (199)$$

or, alternatively, as the difference of upper and lower percentiles divided by the mean value,

$$\frac{\xi_{(1+\eta/100)/2}^+ - \xi_{(1-\eta/100)/2}^-}{\langle \text{Max}(|E_{R,\alpha}|) \rangle} = \frac{\sqrt{\ln \left\{ 1 - \left[\frac{1}{2} \left(1 + \frac{\eta}{100} \right) \right]^{\frac{1}{N}} \right\}^{-1}} - \sqrt{\ln \left\{ 1 - \left[\frac{1}{2} \left(1 - \frac{\eta}{100} \right) \right]^{\frac{1}{N}} \right\}^{-1}}}{\sqrt{0.5772 + \ln(N+1) - \frac{1}{2(N+1)}}}. \quad (200)$$

These definitions yield 1 (0 dB) and 0 ($-\infty$ dB) in the limit $N \rightarrow +\infty$, respectively. They are intercompared in Fig. 20 as a function of N for selected values of the confidence level η (see also [35]).

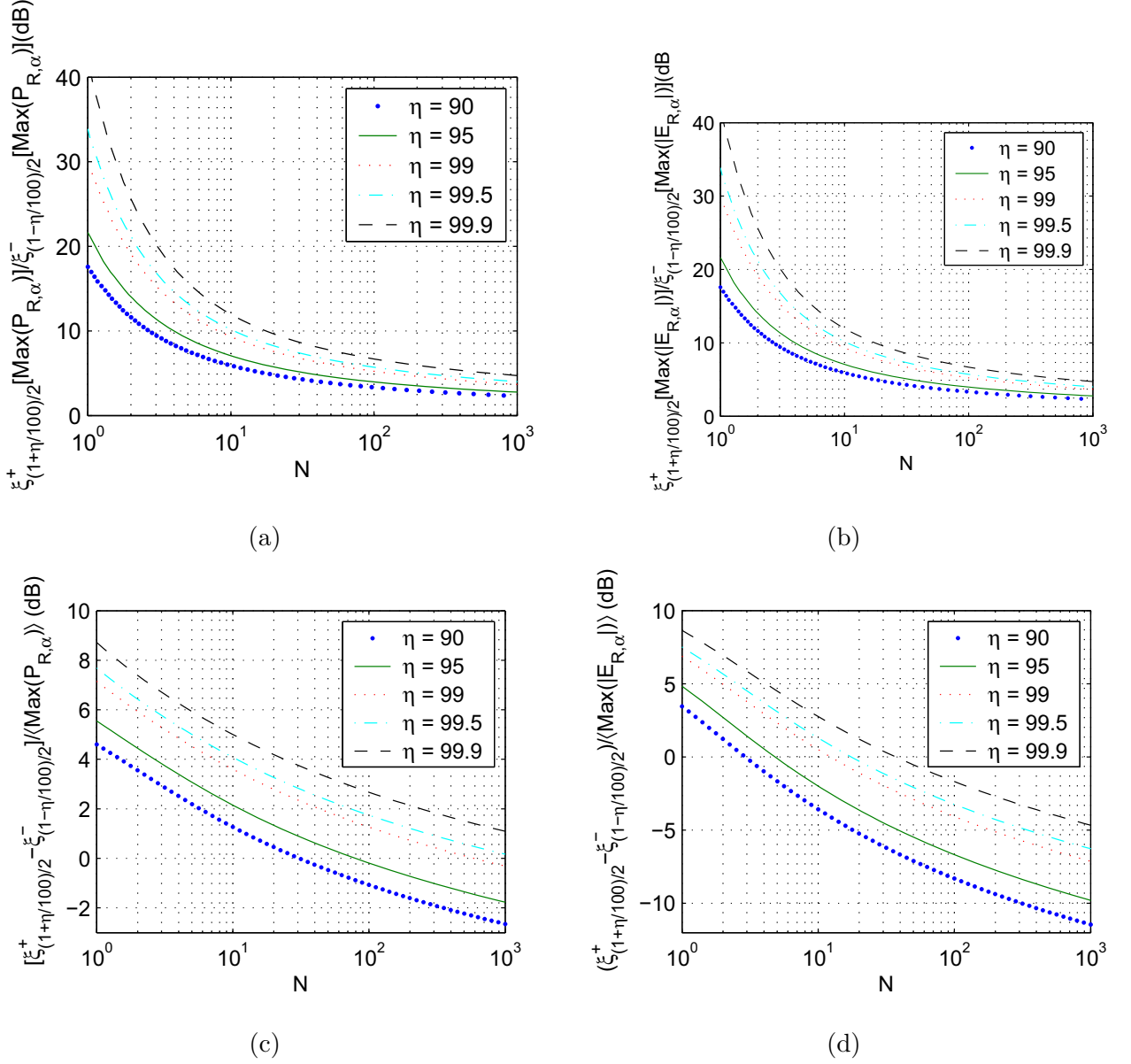


Figure 20: Number of independent stirrer positions N required for obtaining a specified $\eta\%$ confidence interval for (a)(c) $\text{Max}(P_{R,\alpha})$ and (b)(d) $\text{Max}(|E_{R,\alpha}|)$: (a)(b) for interval width defined as ratio of boundaries, $[\xi_{(1+\eta/100)/2}^+ / \xi_{(1-\eta/100)/2}^-]$, as defined by (199); (c)(d) for interval width defined as normalized difference of boundaries, $[\xi_{(1+\eta/100)/2}^+ - \xi_{(1-\eta/100)/2}^-] / \langle \text{Max}(\cdot) \rangle$, as defined by (200).

5.3 Sample Mean, Standard Deviation, and Their Confidence Intervals for Maximum Power or Field Strength

5.3.1 General Considerations

In practice, only comparatively few sample values of the maximum power or field strength are available from experiments in a MT/MSRC, because each new value requires generating a new and sufficiently large set of samples (new tuner sweep), thus involving multiple stirring mechanisms or repeated measurements for different configurations at additional expense. For example, each stir sweep obtained for a single paddle wheel and a single pair of transmitting/receiving antennas produces just one sample value of the ensemble of all maximums for sample sets of a given size N . By placing the transmitting or receiving antenna at $M-1$ other pairs of positions, separated by at least half a wavelength, yields a set of M sample values of the maximum (Fig. 21).

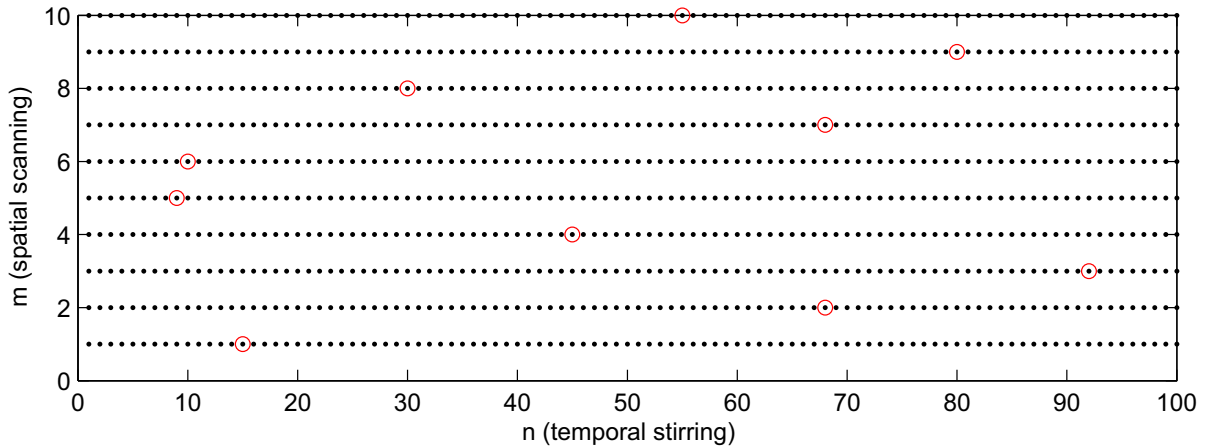


Figure 21: Sampling plane for a two-dimensional space-time (or multiple-stirring) hybrid stirring process, for $N = 100$ stir states and $M = 10$ spatial locations (or secondary stir states). The black dots represent all sample points; the red circles represent M extracted values in hierarchical sampling, e.g., sample maximum values. This example shows secondary sampling as a process of selecting one (n th) primary sample, as, e.g., in maximum-value estimation. More generally, each secondary sample may be extracted as a general function of all primary sample values within the m th row (e.g., sample average, which involves all primary sample values).

Typically, for hybrid stirring in MT/MSRCs, the value of M has to be limited to small values (e.g., $M \leq 10$), while N is often much larger, typically ($N \geq 100$), although the order of magnitude of M and N depends greatly on the level of automation and time (cost) involved in the overall data acquisition. Nevertheless, near the lowest usable frequency, or under other circumstances (e.g., large volum fractional loading of the chamber), N can be relatively small as well. This limits the accuracy of the estimation of ensemble parameters (in particular, mean and standard deviation of the power or

field strength itself) on which the estimation of the non-normalized maximum value and its statistics relies.

The question arises whether $\eta\%$ confidence intervals of the mean value and standard deviation of the *maximum* value can be established when the parameters of their sampling distributions have only limited accuracy (as a result of limited sampling effort) and, if so, how large and representative such intervals are, compared to those obtained when the distribution parameters are known exactly. In other cases, e.g., when the sample statistic of interest is the sample *average* value, the question is whether and how the use of multiple data sets may help in reducing the uncertainty of the sample average value still further.

To address these issues, we shall derive improved confidence intervals that take these accuracy limitations into account. The exact analysis for power or field strength and their sample statistics is difficult, unlike for the Gaussian complex analytic field for which an exact solution can be formulated based on the Student t distribution. We introduce different levels of approximation and sophistication to yield asymptotically exact solutions to the problem of estimating the IFU of the maximum field value from multiple data sets. Explicit calculations will be limited to the simplest cases of improvement. A more elaborated approach, valid for general (including relatively small) values of M and/or N , results in F distributions. These are investigated in Appendix A.

To obtain more accurate confidence intervals, we exploit the fact that, for a *sufficiently large* (but necessarily finite) and fixed value of N for each of the M sample sets, the ensemble distribution of the limit value of the average or maximum value (i.e., for $M \rightarrow +\infty$) is approximately Gauss normal, on account of the central limit theorem. Therefore, we can again use the results of exact sampling theory, as in Sec. 3.

5.3.2 Discretized Hybrid Stirring as Spatio-Temporal Processes or Multi-Dimensional Temporal Sampling Processes

With regard to the possible values of M , the evaluation of field or power for a single mechanical stirring mechanism at a single configuration (i.e., location in space, frequency, antenna placement and orientations, etc.) corresponds to $M = 1$. As will be shown below, this results correctly in confidence intervals of infinite width *when viewed as a random process in space, frequency, antenna placement and orientations, etc., in addition to randomness originating from the single mechanical stirring process*, because of the presence of a factor $M-1$ in the denominator. In actuality, sample statistics for a process with $M - 1 = 0$ degrees of freedom are indeterminate and cannot be analyzed in this manner, i.e., by using sampling theory for estimation of the statistics in a higher dimension. When $M = 1$, the

above methods and results in previous sections for ensemble statistics of a one-dimensional stirring process should be applied.

Alternatively, when multiple stirring mechanisms are applied *that can be considered statistically equivalent*, for which the M marginal stir sweep data sets can be combined into a single superset of size MN without distinction, we can use the above results with N less than or equal to the product of the respective number of independent stir states per stir mechanism, N_i ; cf. (168). This gives rise to similar, but slightly different results with regard to the dependence of the confidence interval on N , compared to that for N_1 and $M \equiv N_2$.

The results in this Section include the case where $M_{\max}$ or $N_{\max}$ (i.e., the maximum possible number of statistically independent sample sets that can be generated) is finite. The extended results for this case follow by increasing M or N by the factor $(M_{\max} - 1)/(M_{\max} - M)$ or $(N_{\max} - 1)/(N_{\max} - N)$, respectively, as treated in Sec. 4.2.2.

5.3.3 Sample Average

5.3.3.1 Power We shall focus attention on the case of the maximum received power $\text{Max}(P_R)$. However, because all powers and field strengths – whether transmitted or received, vectorial or Cartesian – exhibit approximately Gauss normal distributions for their maximum value, the results are applicable to the maximum value of other field quantities as well.

For asymptotically large⁴⁹ values of N , but for general (including relatively small) values of M , it follows from the theory of exact sampling that $\text{Avg}[\text{Max}(P_R)]$ exhibits a Student t distribution with $M-1$ degrees of freedom, i.e.,

$$f_{\text{Avg}[\text{Max}(P_R)]}(\text{avg}[\text{Max}(P_R)]) = \frac{\Gamma(M/2)}{\sqrt{(M-1)\pi} \Gamma[(M-1)/2]} \left(1 + \frac{\{\text{avg}[\text{Max}(P_R)]\}^2}{M-1}\right)^{-M/2}. \quad (201)$$

A 95 % confidence interval for the (ensemble) expected value of $\text{Max}(P_R)$ is

$$\text{avg}[\text{Max}(P_R)] \left(1 + t_{0.025}(M-1) \frac{s_{\text{Avg}[\text{Max}(P_R)]}}{\text{avg}[\text{Max}(P_R)]}\right) \leq \langle \text{Max}(P_R) \rangle \leq \text{avg}[\text{Max}(P_R)] \left(1 + t_{0.975}(M-1) \frac{s_{\text{Avg}[\text{Max}(P_R)]}}{\text{avg}[\text{Max}(P_R)]}\right) \quad (202)$$

or equivalently⁵⁰, with the aid of (3),

$$\text{avg}[\text{Max}(P_R)] \left(1 + \frac{t_{0.025}(M-1) s_{\text{Max}(P_R)}}{\sqrt{M} \text{avg}[\text{Max}(P_R)]}\right) \leq \langle \text{Max}(P_R) \rangle \leq \text{avg}[\text{Max}(P_R)] \left(1 + \frac{t_{0.975}(M-1) s_{\text{Max}(P_R)}}{\sqrt{M} \text{avg}[\text{Max}(P_R)]}\right), \quad (203)$$

in which $t_{0.025}(M-1)$ and $t_{0.975}(M-1) \equiv -t_{0.025}(M-1)$ are the 2.5 % and 97.5 % percentiles (critical values, confidence coefficients), respectively, for a t distribution with $M-1$ degrees of freedom, where

⁴⁹For $N \rightarrow +\infty$, the maximum values are samples from a Gauss normal ensemble distribution of maximums, because χ^2_{2pN} and its associated maximum-value distribution satisfy the central limit theorem.

⁵⁰The maximum value (itself a sample statistic) taken from an infinite set ($M \rightarrow +\infty$) of sample sets, each of size N , exhibits an N -dependent ensemble standard deviation $\sigma_{\text{Max}(P_R)}$. Its sample values for finite M exhibit the sample standard deviation $s_{\text{Max}(P_R)}$.

we have explicitly denoted the number of degrees of freedom of the t distribution for quantile q as $t_q(M-1)$. Numerical values of $t_{0.975}(M-1)$ and $t_{0.995}(M-1)$ at selected values of $M-1$ are tabulated in Appendix B and show that $t_q(M-1)$ is a decreasing function of $M-1$. Pertinently, the width of the interval (203) first decreases faster than $1/\sqrt{M}$, but stabilizes asymptotically to the rate $1/\sqrt{M}$ as M approaches infinity.

Figure 22 compares the M -dependence of the prefactor for the mean-normalized full width of the confidence interval of $\text{Avg}[\text{Max}(P_R)]$ for 95 % and 99 % confidence levels, i.e., from (203) the factor $2 t_{(1+\eta/100)/2}(M-1)/\sqrt{M}$ in

$$\frac{\xi_{(1+\eta/100)/2}^+ - \xi_{(1-\eta/100)/2}^-}{\langle \text{Max}(P_R) \rangle} = \frac{2 t_{(1+\eta/100)/2}(M-1) \sigma_{\text{Max}(P_R)}}{\sqrt{M} \langle \text{Max}(P_R) \rangle} = \frac{2 t_{(1+\eta/100)/2}(M-1)}{\sqrt{M}} \nu_{\text{Max}(P_R)} \quad (204)$$

$$\simeq \frac{2 t_{(1+\eta/100)/2}(M-1)}{\sqrt{M}} n_{\text{Max}(P_R)} \quad (205)$$

for $\eta = 95$ and 99 , where $\nu_{\text{Max}(P_R)}$ depends on N . For example, for $M = 12$, we have $2 t_{0.975}(11)/\sqrt{12} = 1.2702$ compared to $2 t_{0.995}(11)/\sqrt{12} = 1.7956$.

For the Cartesian power, we obtain from (204) with (192) and (196) the asymptotic⁵¹ relative full width ($N \rightarrow +\infty$) as

$$\frac{2 t_{(1+\eta/100)/2}(M-1) \sigma_{\text{Max}(P_{R,\alpha})}}{\sqrt{M} \langle \text{Max}(P_{R,\alpha}) \rangle} \rightarrow \frac{2 t_{(1+\eta/100)/2}(M-1)}{\sqrt{M}} \frac{\pi}{\sqrt{6} [0.5772 + \ln(N)]}. \quad (206)$$

Since $\sqrt{x} > \ln(x)$ and with $t_q(x)$ being a monotonically decreasing function of x , it follows from (206) that the confidence interval for the expectation value of the maximum Cartesian power becomes narrower at a faster rate when M increases than when N increases. Consequently, with respect to the reduction of IFU for the maximum power, more gains are made from increasing the *number* of relatively short stir sequences (as achieved through spatial scanning or applying different stirrers or stirring methods) than from increasing the *length* of a single stir sequence, for example by using a smaller stir increment (assuming that the latter reduction still increases the number of independent samples N , i.e., that it does not markedly increase the correlation coefficient). The result is an artefact of hierarchical sampling for the maximum value.

For finite values of M_{max} , the distribution of $\text{Avg}[\text{Max}(P_R)]$ is only approximatively given by the Student t distribution (201). Figure 23 shows the mean-normalized width of a 95 % confidence interval for $\langle \text{Max}(P_R) \rangle$ as a function of M ($\leq M_{\text{max}}$) for finite values of M_{max} . It is observed that, for arbitrary M , a smaller value of M_{max} gives rise to a narrower confidence interval, especially when M approaches M_{max} . Thus, uncertainties evaluated on the basis of $M_{\text{max}} \rightarrow +\infty$ can be considered to be conservative (worst-case) with regard to the realistic case of finite M_{max} .

⁵¹In (206), the term 0.5772 has been retained because of the relatively slow increase of $\ln(N)$ toward infinity.

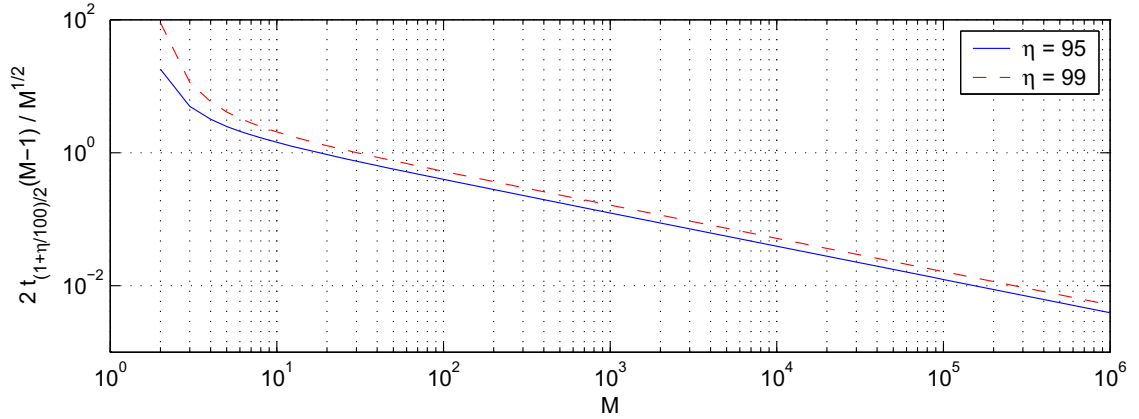


Figure 22: Multiplier for full width (normalized with respect to $\text{avg}[\text{Max}(P_R)]$) of 95 % and 99 % confidence intervals for the expected value of the maximum received power, $\langle \text{Max}(P_R) \rangle$, as a function of the number of generated independent stir sequences M (number of degrees of freedom plus one), for an unlimited ensemble size (i.e., maximum possible number) of stir sequences ($M_{\text{max}} \rightarrow +\infty$).

To demonstrate the combined dependence of the width of the confidence interval on both M and N , Figure 24 shows contour plots of the full width of 95 % and 99 % confidence intervals (203) as a function of $\log_{10}(M)$ and $\log_{10}(N)$. Increasing M is seen to have a more profound effect on reducing the interval width than N has, particularly for large N . This shows that increasing the number of degrees of freedom has a more profound effect on sampling statistics than on ensemble statistics.

Figures 25 and 26 show mean-normalized boundaries and widths of 95 %, 99 %, and 99.5 % confidence intervals for the expected value of the maximum power. Like for confidence intervals for the average value in Sec. 3.1.1, the Gauss normal approximation is accurate with regard to the estimation of the width of the interval, but less so for the location at relatively small values of N .

5.3.3.2 Field Strength A similar analysis applies to the sample mean value of the field strength, because χ_{2pN} like χ_{2pN}^2 approaches asymptotic Gauss normality when $N \rightarrow +\infty$. For finite N , the asymptotic results for χ_{2pN} are approached even more rapidly when increasing N than for χ_{2pN}^2 . Thus, expressions (203) and (205) remain valid, mutatis mutandis. Compared to (192), the coefficient of variation for $\text{Max}(|E_{R,\alpha}|)$ in (205) differs by a factor 1/2 but maintains the same dependence on N as for $\text{Max}(P_{R,\alpha})$, viz.,

$$\nu_{\text{Max}(|E_{R,\alpha}|)} \simeq \frac{\sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}}{2 \left[0.5772 + \ln(N+1) - \frac{1}{2(N+1)} \right]}. \quad (207)$$

⁵²Strictly, the results in this Section apply to $N \rightarrow +\infty$ only. However, for finite N , the maximum-value distributions for single stir sequences rapidly approach Gauss normality when N is increased. Numerical results for finite N are included here to demonstrate the rate of convergence in practical cases to interval boundaries and widths for $N \rightarrow +\infty$.

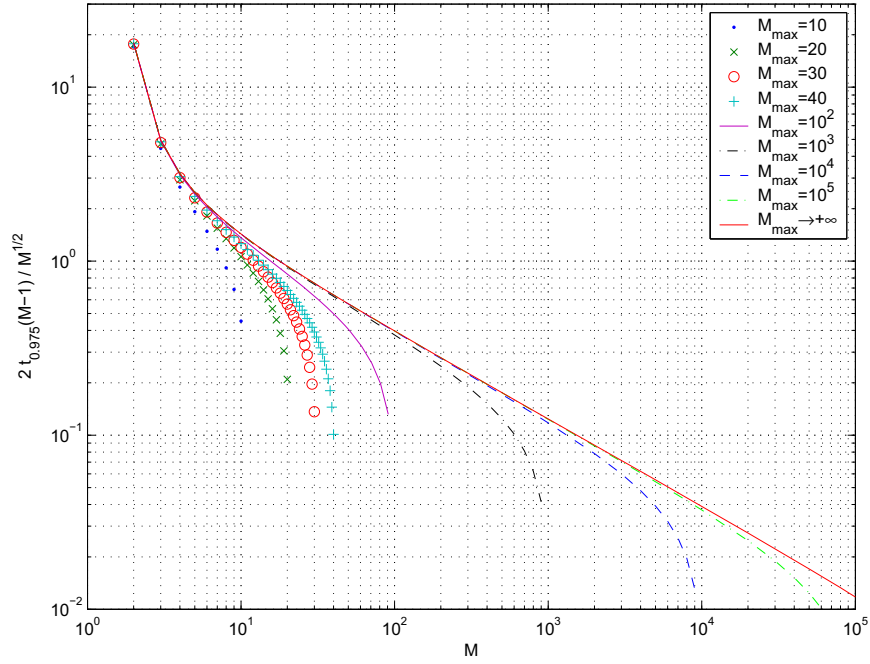


Figure 23: Multiplier for full width (normalized with respect to $\text{avg}[\text{Max}(P_R)]$) of 95 % confidence intervals for the expected value of the maximum received power, $\langle \text{Max}(P_R) \rangle$, as a function of the number of generated independent stir sequences, M (number of degrees of freedom plus one), for a limited ensemble size (maximum possible number) of stir sequences ($M_{\text{max}} < +\infty$).

As a result, the confidence interval for the sample mean of the maximum field strength is half as wide as for the power when expressed in linear units, although of equal width when expressed in dB.

Figures 27 and 28 show mean-normalized boundaries and widths of 95 %, 99 % , and 99.5 % confidence intervals for the expected value of the maximum field strength.

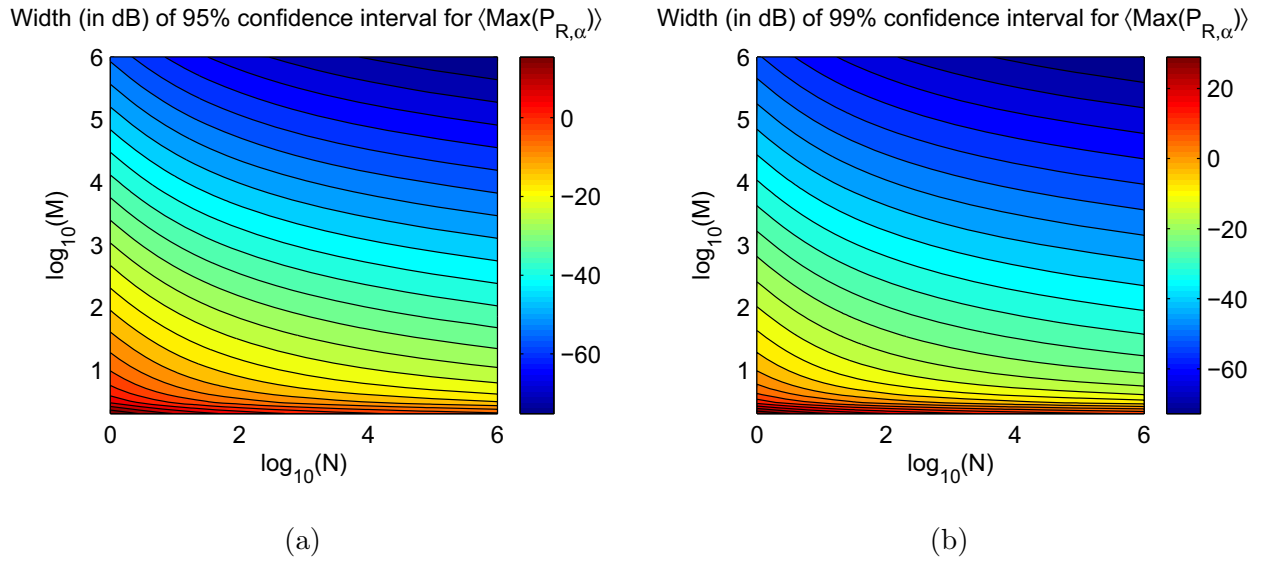


Figure 24: Full width (normalized with respect to $\langle P_{R,\alpha} \rangle$) of (a) 95 % and (b) 99 % confidence intervals for the average value of the maximum received power, $\langle \text{Max}(P_{R,\alpha}) \rangle$, as a function of the number of selected independent stir sequences M and the number of independent stir states per sequence N . The mean-normalized width (in dB) for the confidence intervals for $\langle \text{Max}(|E_{R,\alpha}|) \rangle$ is the same as for $\langle \text{Max}(P_{R,\alpha}) \rangle$.

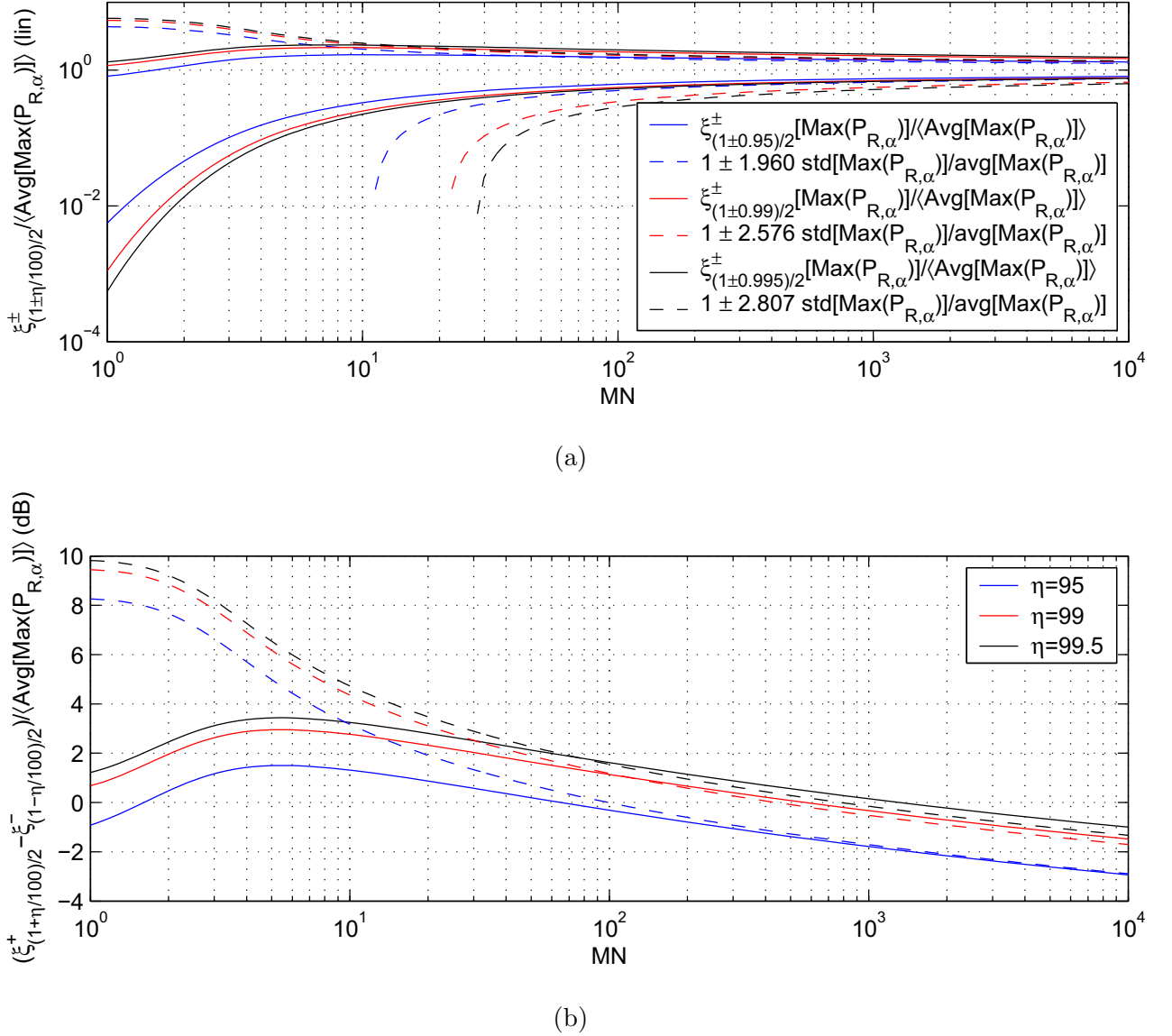
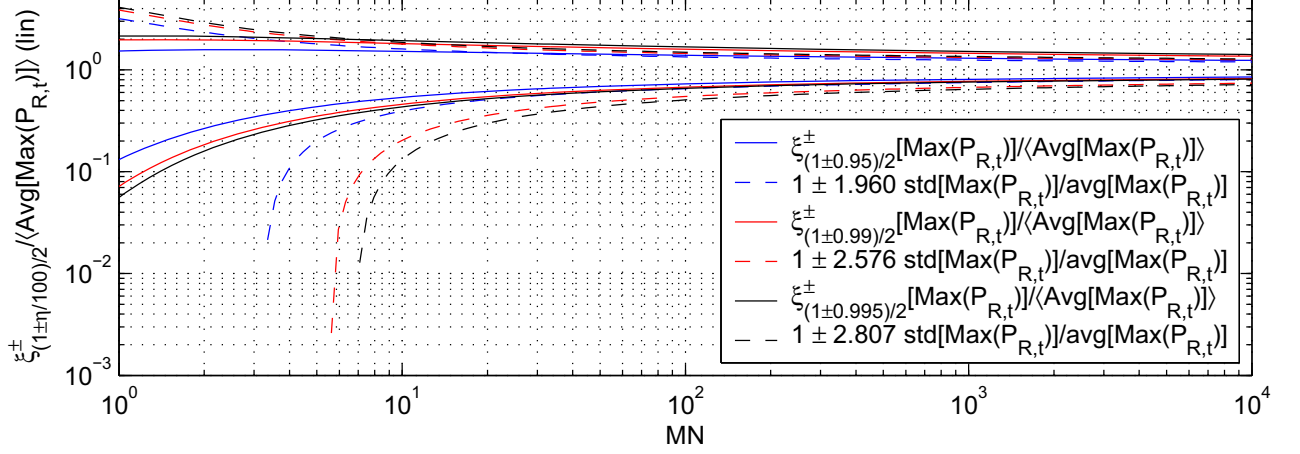
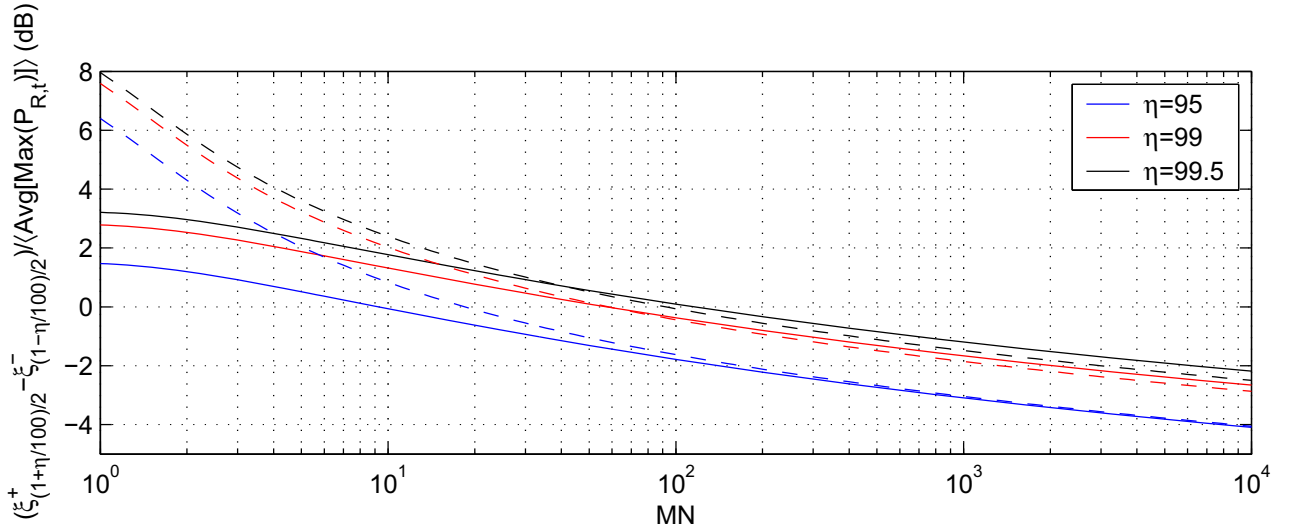


Figure 25: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Max}(P_{R,\alpha})$ ($p = 1$): (solid lines) exact, based on percentiles of maximum-value distribution of $P_{R,\alpha}$ (χ^2_{2N} parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha})), 1 + 1.960 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha}))]$, $[1 - 2.576 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha})), 1 + 2.576 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha}))]$, and $[1 - 2.807 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha})), 1 + 2.807 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha}))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.



(a)



(b)

Figure 26: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Max}(P_{R,t})$ ($p = 3$): (solid lines) exact, based on percentiles of maximum-value distribution of $P_{R,t}$ (χ_{2N}^2 parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Max}(P_{R,t})}/\text{avg}(\text{Max}(P_{R,t})), 1 + 1.960 s_{\text{Max}(P_{R,t})}/\text{avg}(\text{Max}(P_{R,t}))]$, $[1 - 2.576 s_{\text{Max}(P_{R,t})}/\text{avg}(\text{Max}(P_{R,t})), 1 + 2.576 s_{\text{Max}(P_{R,t})}/\text{avg}(\text{Max}(P_{R,t}))]$, and $[1 - 2.807 s_{\text{Max}(P_{R,t})}/\text{avg}(\text{Max}(P_{R,t})), 1 + 2.807 s_{\text{Max}(P_{R,t})}/\text{avg}(\text{Max}(P_{R,t}))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.

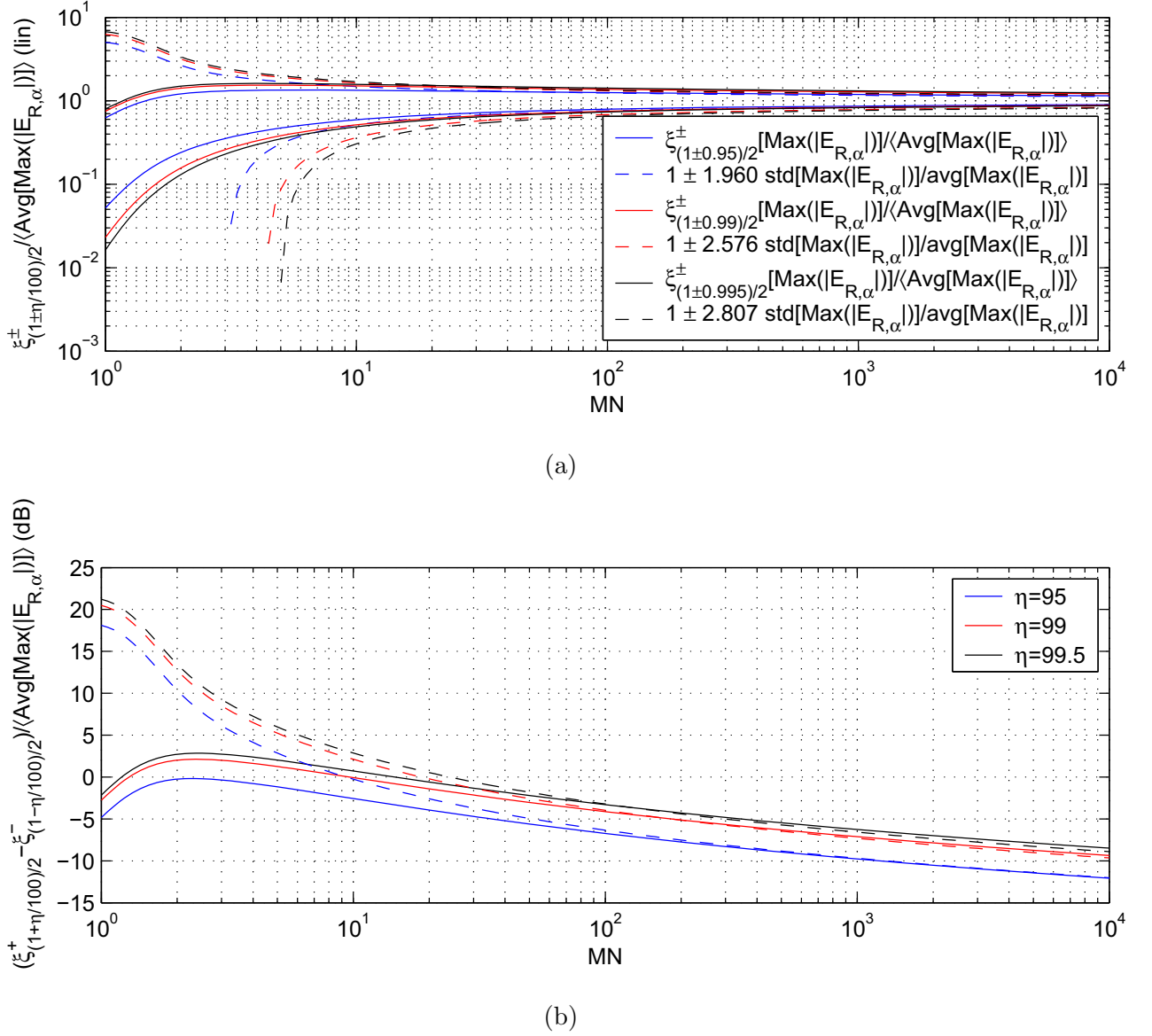


Figure 27: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Max}(|E_{R,\alpha}|)$ ($p = 1$): (solid lines) exact, based on percentiles of maximum-value distribution of $|E_{R,\alpha}|$ (χ^2_{2N} parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 \ s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|)), 1 + 1.960 \ s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|))]$, $[1 - 2.576 \ s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|)), 1 + 2.576 \ s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|))]$, and $[1 - 2.807 \ s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|)), 1 + 2.807 \ s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.

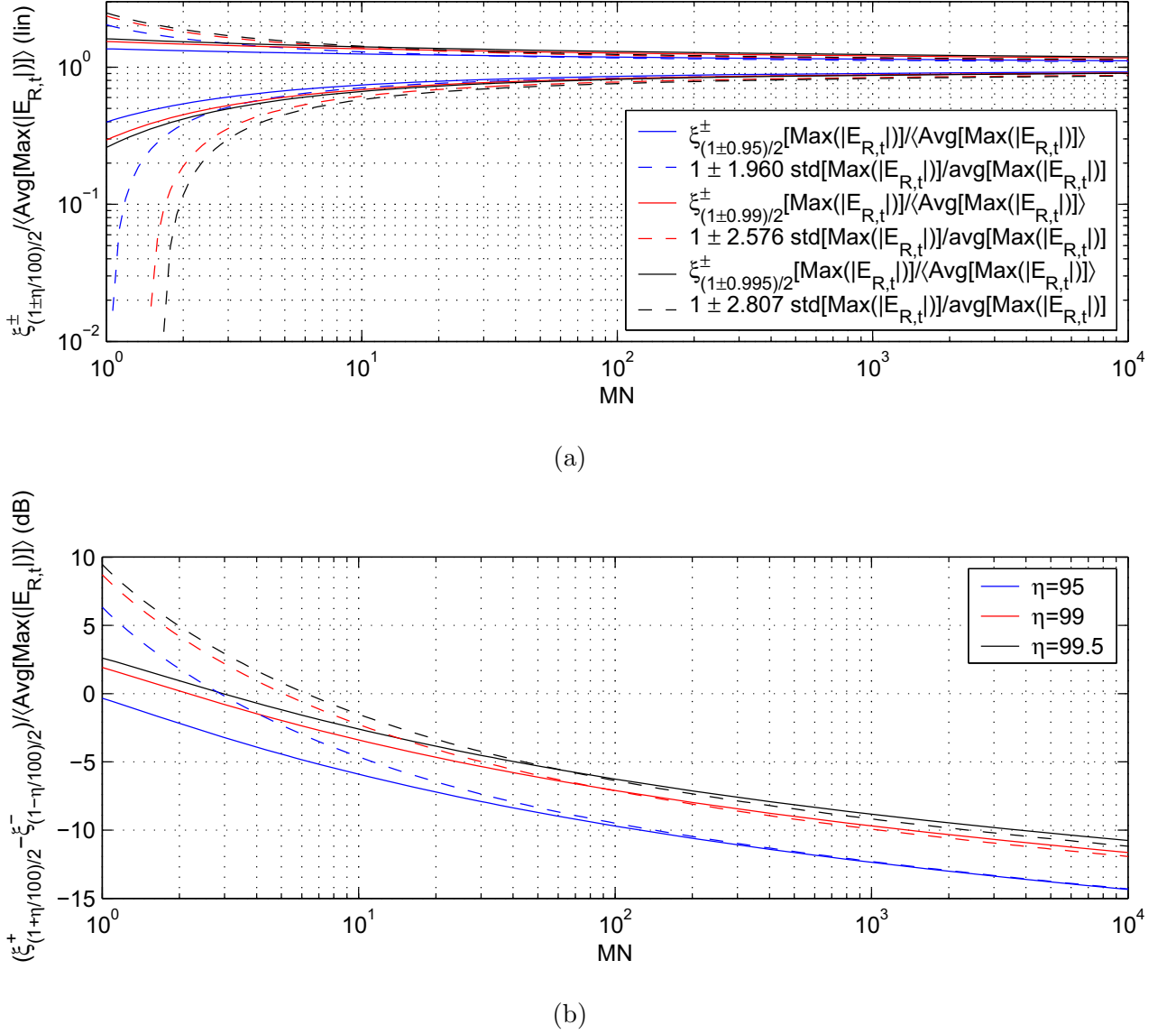


Figure 28: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Max}(|E_{R,t}|)$ ($p = 3$): (solid lines) exact, based on percentiles of maximum-value distribution of $|E_{R,t}|$ (χ^2_{2N} parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 \text{ std}[\text{Max}(|E_{R,t}|)] / \text{avg}(\text{Max}(|E_{R,t}|)), 1 + 1.960 \text{ std}[\text{Max}(|E_{R,t}|)] / \text{avg}(\text{Max}(|E_{R,t}|))]$, $[1 - 2.576 \text{ std}[\text{Max}(|E_{R,t}|)] / \text{avg}(\text{Max}(|E_{R,t}|)), 1 + 2.576 \text{ std}[\text{Max}(|E_{R,t}|)] / \text{avg}(\text{Max}(|E_{R,t}|))]$, and $[1 - 2.807 \text{ std}[\text{Max}(|E_{R,t}|)] / \text{avg}(\text{Max}(|E_{R,t}|)), 1 + 2.807 \text{ std}[\text{Max}(|E_{R,t}|)] / \text{avg}(\text{Max}(|E_{R,t}|))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.

5.3.4 Sample Standard Deviation

5.3.4.1 Power To obtain an $\eta\%$ confidence interval for the standard error of $\text{Max}(P_R)$, we rely again on the fact that, for a sufficiently large size N of each individual sample set, the ensemble distribution for M sampled maximum values is approximately Gauss normal, irrespective whether M is large or small.

The normalized sample variance of the maximum-value distribution, $X \triangleq (\sqrt{M-1} s_{\text{Max}(P_R)} / \sigma_{\text{Max}(P_R)})^2$, exhibits a standard χ^2 pdf with $M-1$ degrees of freedom, i.e.,

$$f_X(x) = \frac{x^{(M-3)/2} \exp(-x/2)}{2^{(M-1)/2} \Gamma[(M-1)/2]}, \quad (208)$$

with a (two-sided) 95 % confidence interval

$$\chi_{0.025}^2(M-1) \leq X \leq \chi_{0.975}^2(M-1) \quad (209)$$

in which $\chi_q^2(M-1)$ denote percentiles for $q = 2.5\%$ and 97.5% for this pdf. Numerical values of $\chi_{0.995}^2$, $\chi_{0.975}^2$, $\chi_{0.025}^2$, and $\chi_{0.005}^2$ at selected values of $M-1$ are tabulated in Appendix C. Hence, from (209) and $\sqrt{\chi_q^2(M-1)} = \chi_q(M-1)$,

$$\frac{s_{\text{Max}(P_R)} \sqrt{M-1}}{\chi_{0.975}(M-1)} \leq \sigma_{\text{Max}(P_R)} \leq \frac{s_{\text{Max}(P_R)} \sqrt{M-1}}{\chi_{0.025}(M-1)}. \quad (210)$$

Now, for Cartesian components, with (183) and (188),

$$\sigma_{\text{Max}(P_{R,\alpha})} = \sigma_{P_{R,\alpha}} \sqrt{\sum_{i=1}^N i^{-2}} \simeq \sigma_{P_{R,\alpha}} \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}, \quad (211)$$

showing that $\sigma_{P_{R,\alpha}}$ may serve as a standardization factor for the boundaries of the confidence interval of $\sigma_{\text{Max}(P_{R,\alpha})}$. Hence, in terms of the measured value of the standard error $s_{P_{R,\alpha}}$, a 95% confidence interval is

$$\frac{s_{P_{R,\alpha}} \sqrt{M-1}}{\chi_{0.975}(M-1)} \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}} \leq \sigma_{\text{Max}(P_{R,\alpha})} \leq \frac{s_{P_{R,\alpha}} \sqrt{M-1}}{\chi_{0.025}(M-1)} \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}. \quad (212)$$

Figures 29 and 30 show limits and widths of 95 % and 99 % confidence intervals for $\sigma_{\text{Max}(P_{R,\alpha})}$, normalized by $\sigma_{P_{R,\alpha}}$, as a function of $\log_{10}(M)$ and $\log_{10}(N)$.

As a complement to the confidence interval for $\langle \text{Max}(P_{R,\alpha}) \rangle$ in Sec. 5.3.3.1, the corresponding interval for the associated $s_{\text{Avg}[\text{Max}(P_{R,\alpha})]} = \sigma_{\text{Max}(P_{R,\alpha})} / \sqrt{M}$ follows from (212) immediately as

$$\frac{s_{P_{R,\alpha}}}{\chi_{0.975}(M-1)} \sqrt{\frac{M-1}{M}} \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}} \leq s_{\text{Avg}[\text{Max}(P_{R,\alpha})]} \leq \frac{s_{P_{R,\alpha}}}{\chi_{0.025}(M-1)} \sqrt{\frac{M-1}{M}} \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}. \quad (213)$$

For increasing M , this interval narrows for increasing M at an asymptotic rate $1/\sqrt{M}$, on account of the asymptotic M -dependence of $1/\chi_q(M-1)$ (cf. Appendix C).

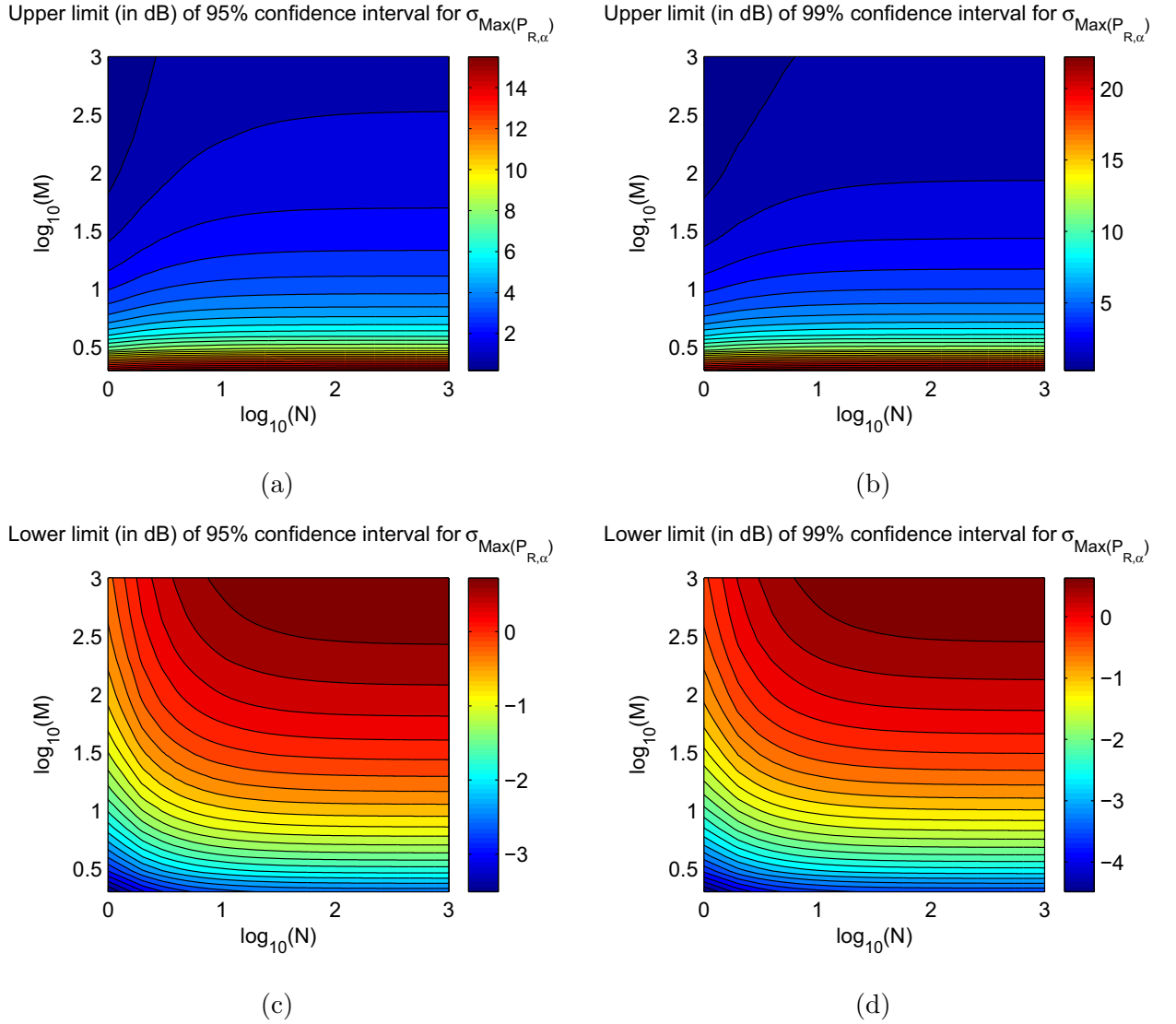


Figure 29: (a)(b) Upper and (c)(d) lower limits (in dB and units $\sigma_{P_{R,\alpha}}$) of (a)(c) 95% and (b)(d) 99% confidence intervals for the standard deviation of the maximum received power, $\sigma_{\text{Max}(P_{R,\alpha})}$, as a function of the specified number of selected independent stir sequences M and the number of independent stir states per sequence N .

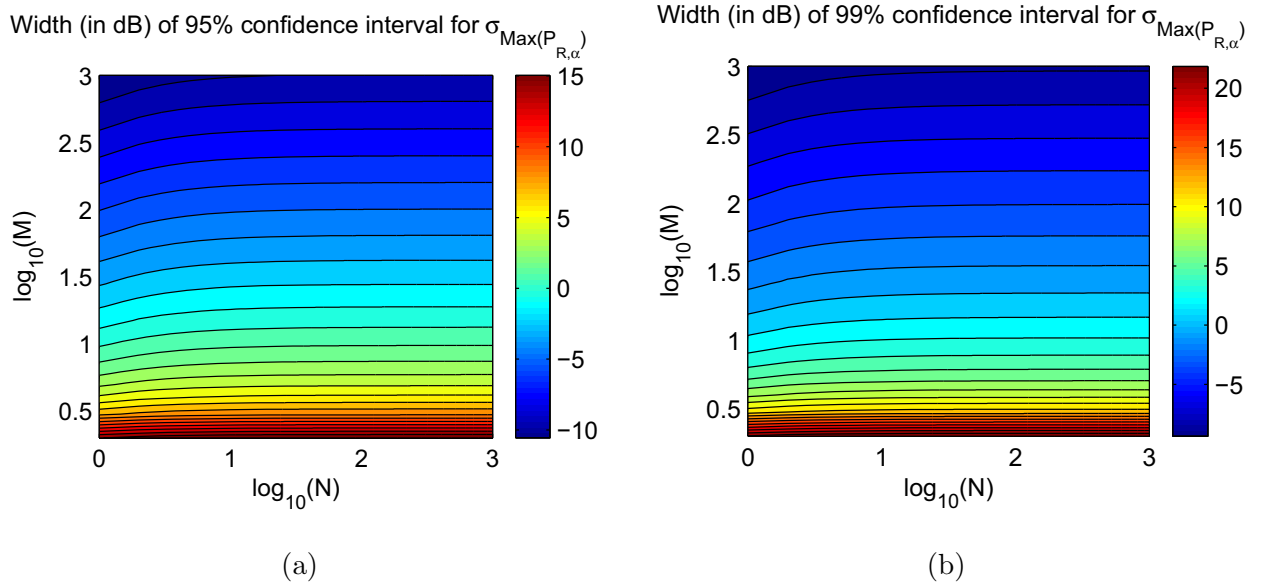


Figure 30: Widths (in dB and units $\sigma_{P_{R,\alpha}}$) of (a) 95 %- and (b) 99 % confidence intervals for the standard deviation of the maximum received power, $\sigma_{\text{Max}(P_{R,\alpha})}$, as a function of the specified number of selected independent stir sequences M and the number of independent stir states per sequence N .

5.3.4.2 Field Strength Again, the results are derived in a similar way to those for the power. However, the functional dependence of $\sigma_{\text{Max}(|E_{R,\alpha}|)}$ on N , given by (194), is different from that for $\sigma_{\text{Max}(P_{R,\alpha})}$ in (188). This difference affects the dependence of the limits and width of the confidence interval on N . In general, similar to (210),

$$\frac{s_{\text{Max}(|E_R|)}\sqrt{M-1}}{\chi_{0.975}(M-1)} \leq \sigma_{\text{Max}(|E_R|)} \leq \frac{s_{\text{Max}(|E_R|)}\sqrt{M-1}}{\chi_{0.025}(M-1)}. \quad (214)$$

For Cartesian components, using (185) and (194),

$$\frac{s_{|E_{R,\alpha}|}\sqrt{M-1}}{\chi_{0.975}(M-1)} \frac{\sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}}{\sqrt{(4-\pi) \left[0.5772 + \ln(N+1) - \frac{1}{2(N+1)}\right]}} \leq \sigma_{\text{Max}(|E_{R,\alpha}|)} \leq \frac{s_{|E_{R,\alpha}|}\sqrt{M-1}}{\chi_{0.025}(M-1)} \frac{\sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}}{\sqrt{(4-\pi) \left[0.5772 + \ln(N+1) - \frac{1}{2(N+1)}\right]}}. \quad (215)$$

Figures 31 and 32 show limits and widths of 95% and 99% confidence intervals for $\sigma_{\text{Max}(|E_{R,\alpha}|)}$ normalized by $\sigma_{|E_{R,\alpha}|}$ as a function of $\log_{10}(M)$ and $\log_{10}(N)$. As expected, the graphs are identical with those for $\sigma_{\text{Max}(P_{R,\alpha})}$ normalized by $\sigma_{P_{R,\alpha}}$, as a consequence of the relationship between $\chi^{(2)}$ parent distributions of standardized P_R and $|E_R|$ by $\sigma_{P_{R,\alpha}}$ and $\sigma_{|E_{R,\alpha}|}$, respectively. In absolute terms, however (i.e., if not standardized), quantitative and qualitative differences exist between corresponding graphs for $\text{Max}(|E_{R,\alpha}|)$ and $\text{Max}(P_{R,\alpha})$. For example, the non-normalized width of the interval for $\sigma_{\text{Max}(|E_{R,\alpha}|)}$ shows a marginal *increase* with increasing N , owing to an increase of the upper limit of this interval. After normalization, a proper decrease of both the interval width and the upper interval limit is retrieved.

Finally, the confidence interval for the standard error $s_{\text{Avg}[\text{Max}(|E_{R,\alpha}|)]} = \sigma_{\text{Max}(|E_{R,\alpha}|)}/\sqrt{M}$ again follows from (215) as

$$\frac{s_{|E_{R,\alpha}|}}{\chi_{0.975}(M-1)} \frac{\sqrt{\frac{M-1}{M}} \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}}{\sqrt{(4-\pi) \left[0.5772 + \ln(N+1) - \frac{1}{2(N+1)}\right]}} \leq s_{\text{Avg}[\text{Max}(|E_{R,\alpha}|)]} \leq \frac{s_{|E_{R,\alpha}|}}{\chi_{0.025}(M-1)} \frac{\sqrt{\frac{M-1}{M}} \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}}{\sqrt{(4-\pi) \left[0.5772 + \ln(N+1) - \frac{1}{2(N+1)}\right]}}. \quad (216)$$

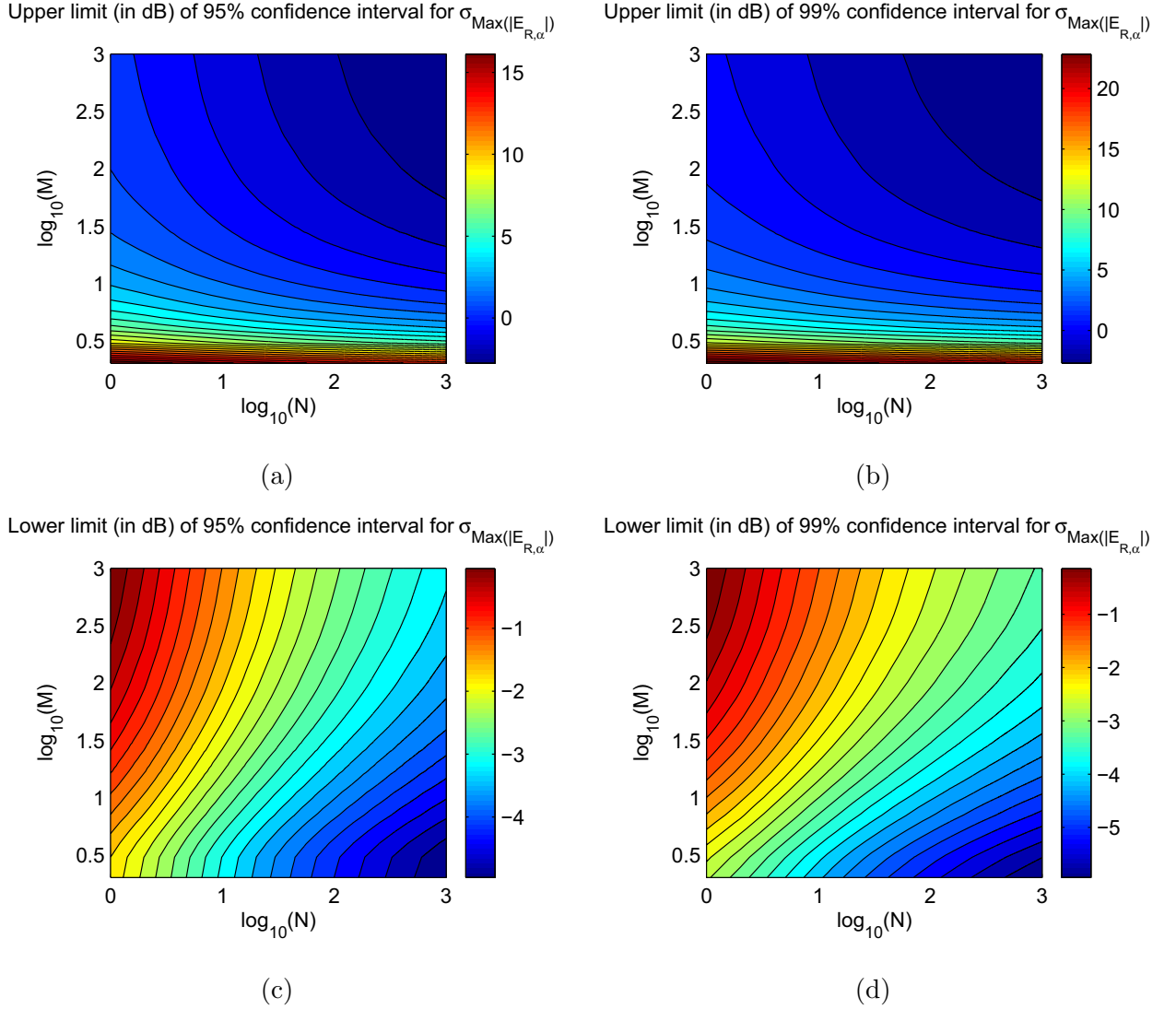


Figure 31: (a)(b) Upper and (c)(d) lower limits (in dB and units $\sigma_{|E_{R,\alpha}|}$) of (a)(c) 95 % and (b)(d) 99 % confidence intervals for the standard deviation of the maximum received field strength, $\sigma_{\text{Max}(|E_{R,\alpha}|)}$, as a function of the specified number of selected independent stir sequences M and the number of independent stir states per sequence N .

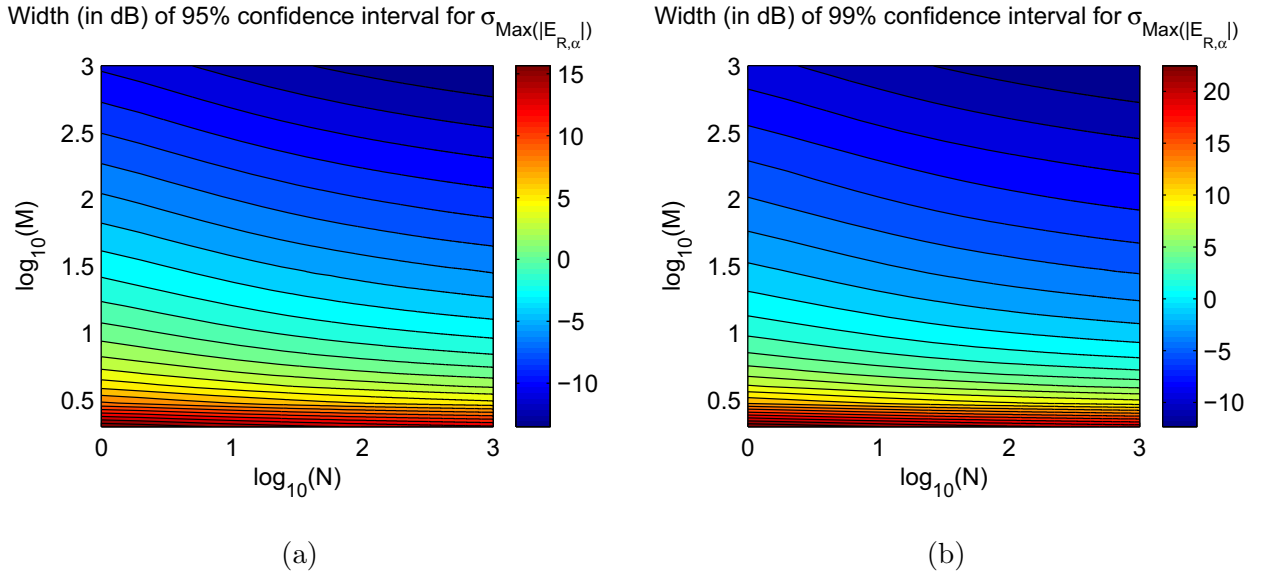


Figure 32: Widths (in dB and units $\sigma_{|E_{R,\alpha}|}$) of (a) 95% and (b) 99% confidence intervals for the standard deviation of the maximum received field strength, $\sigma_{\text{Max}(|E_{R,\alpha}|)}$, as a function of the specified number of selected independent stir sequences M and the number of independent stir states per sequence N .

5.3.5 Application: Uncertainties Associated with Evaluation of Field Uniformity for Validating a MT/MSRC according to IEC 61000-4-21 Ed. 1

As an application, we consider the determination of 95 % confidence intervals for the mean value of the Cartesian maximum field strength for a specific choice of the value of M . Of particular importance are the values⁵³ $M = 8$ and $M = 3 \times 8 = 24$, used in the chamber validation procedure in IEC 61000-4-21 Ed. 1 [2, Annex B]. With the aid of the percentiles listed in Tbl. 6 in Appendix B, but evaluated to higher precision for the present purpose, the 95 % and 99 % confidence interval widths of $\langle \text{Max}(|E_{R,\alpha}|) \rangle$ for $M-1 = 7$ degrees of freedom are

$$\frac{2 t_{0.975}(7)}{\sqrt{8}} = 1.672 (\sim 4.46 \text{ dB}), \quad \frac{2 t_{0.995}(7)}{\sqrt{8}} = 2.474 (\sim 7.87 \text{ dB}) \quad (217)$$

respectively, whereas for $M-1 = 23$,

$$\frac{2 t_{0.975}(23)}{\sqrt{24}} = 0.845 (\sim -1.46 \text{ dB}), \quad \frac{2 t_{0.995}(23)}{\sqrt{24}} = 1.146 (\sim 1.19 \text{ dB}). \quad (218)$$

To estimate the field uniformity, these values are still to be reduced by multiplying with the N -dependent experimental sample coefficient of variation $n_{\text{Max}(|E_{R,\alpha}|)}$, whose corresponding ensemble value is given by (207). Resulting widths of 95 % confidence intervals for $\text{Avg}[\text{Max}(|E_{R,\alpha}|)]$, i.e.,

$$\frac{\xi_{(1+\eta/100)/2}^+ - \xi_{(1-\eta/100)/2}^-}{\langle \text{Max}(|E_{R,\alpha}|) \rangle} = \frac{2 t_{(1+\eta/100)/2}(M-1)}{\sqrt{M}} \frac{\sqrt{\sum_{i=1}^N i^{-2}}}{2 \sum_{i=1}^N i^{-1}} \simeq \frac{t_{(1+\eta/100)/2}(M-1)}{\sqrt{M}} \frac{\sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}}{0.5772 + \ln(N+1) - \frac{1}{2(N+1)}} \quad (219)$$

with $\eta = 95$ are listed in Table 2 for selected values⁵⁴ of N and either value of M .

Table 2: Normalized width (linear; in units $\text{avg}[\text{Max}(|E_{R,\alpha}|)]$) of 95 % confidence interval of mean-normalized $\langle \text{Max}(|E_{R,\alpha}|) \rangle$, for selected values of M and N .

	$N = 10$	$N = 12$	$N = 30$	$N = 100$	$N = 300$	$N = 1\,000$
$\nu_{\text{Max}(E_{R,\alpha})}$	0.213	0.202	0.159	0.123	0.102	0.086
$M = 8$	0.355	0.337	0.266	0.206	0.170	0.143
$M = 24$	0.180	0.170	0.134	0.104	0.086	0.072

Confidence intervals for the standard deviation of the maximum field strength can also be calculated, with the aid of the numerical values in Table 7 in Appendix C: the 95 % and 99 % interval widths of $\sigma_{\text{Max}(|E_{R,\alpha}|)}$ for $M-1 = 7$ are now proportional to

$$\frac{\sqrt{7}}{\chi_{0.025}(7)} - \frac{\sqrt{7}}{\chi_{0.975}(7)} = 1.374 (\sim 2.76 \text{ dB}), \quad \frac{\sqrt{7}}{\chi_{0.005}(7)} - \frac{\sqrt{7}}{\chi_{0.995}(7)} = 2.073 (\sim 6.33 \text{ dB}) \quad (220)$$

⁵³These apply to an 8-point validation procedure. Other standards use 9 or a different number of locations, for which the value of $M-1$ must be adapted accordingly.

⁵⁴Recall that, strictly, the applicability of the analysis in this Section assumes $N \rightarrow +\infty$; only then does the maximum-value distribution reach true Gauss normality.

whereas for $M-1 = 23$,

$$\frac{\sqrt{23}}{\chi_{0.025}(23)} - \frac{\sqrt{23}}{\chi_{0.975}(23)} = 0.625 (\sim -4.08 \text{ dB}), \quad \frac{\sqrt{23}}{\chi_{0.005}(23)} - \frac{\sqrt{23}}{\chi_{0.995}(23)} = 0.855 (\sim -1.36 \text{ dB}). \quad (221)$$

These values again still need to be multiplied by the coefficients in (215) to yield the confidence intervals in case of large but finite N . For easy comparison with the confidence interval for $\langle \text{Max}(|E_{R,\alpha}|) \rangle$, as well as for subsequent calculation of the IEC field nonuniformity metric, the confidence interval for $\sigma_{\text{Max}(|E_{R,\alpha}|)}$ can be normalized to the expected maximum value, rather than to $\sigma_{|E_{R,\alpha}|}$. From (214),

$$\frac{n_{\text{Max}(|E_R|)} \sqrt{M-1}}{\chi_{0.975}(M-1)} \leq \frac{\sigma_{\text{Max}(|E_R|)}}{\langle \text{Max}(|E_R|) \rangle} \leq \frac{n_{\text{Max}(|E_R|)} \sqrt{M-1}}{\chi_{0.025}(M-1)} \quad (222)$$

with $\sigma_{\text{Max}(|E_R|)} / \langle \text{Max}(|E_R|) \rangle \equiv \nu_{\text{Max}(|E_R|)}$, whence of the Cartesian field components, with the aid of (192) and (193),

$$\frac{\sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}}{2 \left[0.5772 + \ln(N+1) - \frac{1}{2(N+1)} \right]} \frac{\sqrt{M-1}}{\chi_{0.975}(M-1)} \leq \frac{\sigma_{\text{Max}(|E_{R,\alpha}|)}}{\langle \text{Max}(|E_{R,\alpha}|) \rangle} \leq \frac{\sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}}{2 \left[0.5772 + \ln(N+1) - \frac{1}{2(N+1)} \right]} \frac{\sqrt{M-1}}{\chi_{0.025}(M-1)}. \quad (223)$$

With this normalization, 95 % intervals for selected values of N and either value of M are given in Table 3, showing that for $12 \leq N \leq 1000$, the relative width of the uncertainty interval for $\sigma_{\text{Max}(|E_{R,\alpha}|)} / \langle \text{Max}(|E_{R,\alpha}|) \rangle$ is between 5 % and 28 %. Comparison with the values in Table 2 shows that the width of the interval for $\sigma_{\text{Max}(|E_{R,\alpha}|)}$ is of the same order of magnitude and only marginally smaller (about 2 to 3 dB, after conversion) than that for $\langle \text{Max}(|E_{R,\alpha}|) \rangle$, for any value of N . This underlines the substantial effect that sample variability may have on the binary pass/fail outcome of the validation procedure and the accuracy of field estimation, particularly when the value of N is smaller than 30: a different choice of M chosen locations or orientations of the field probe may produce a more or less favourable results.

Table 3: Width (linear; in units $\text{avg}[\text{Max}(|E_{R,\alpha}|)]$) of 95 % confidence interval of $\sigma_{\text{Max}(|E_{R,\alpha}|)} / \langle \text{Max}(|E_{R,\alpha}|) \rangle$, for selected values of M and N .

	$N = 10$	$N = 12$	$N = 30$	$N = 100$	$N = 300$	$N = 1\,000$
$\nu_{\text{Max}(E_{R,\alpha})}$	0.213	0.202	0.159	0.123	0.102	0.086
$M = 8$	0.292	0.277	0.218	0.169	0.140	0.118
$M = 24$	0.133	0.126	0.099	0.077	0.064	0.054

Corresponding values of the field nonuniformity coefficient, as quantified by the IEC metric for chamber validation [2, App. B], i.e.,

$$20 \log_{10} \left(1 + \frac{\sigma_{\text{Max}(|E_{R,\alpha}|)}}{\langle \text{Max}(|E_{R,\alpha}|) \rangle} \right) \quad (224)$$

are listed in Table 4 for the case of known distribution parameters ($\mathcal{N} \rightarrow +\infty$). The Table shows that, for example, when using all 3×8 measured maximum values of the Cartesian components of a 3-axis field probe at eight location in an ideal reverberant field, the sampling uncertainty interval for the field uniformity level increases from 0.64 dB to 1.08 dB if the chosen number of independent stir states N is reduced from 100 to 10.

Table 4: Width of uncertainty interval for field nonuniformity level $20 \log_{10}(1 + \sigma_{\text{Max}(|E_{R,\alpha}|)} / \langle \text{Max}(|E_{R,\alpha}|) \rangle)$ (linear and in dB) at 95 % confidence level based on distribution with *a priori* known parameter values ($\mathcal{N} \rightarrow +\infty$).

	$N = 10$	$N = 12$	$N = 30$	$N = 100$	$N = 300$	$N = 1\,000$
$M = 8$	1.292 (2.23 dB)	1.277 (2.12 dB)	1.218 (1.71 dB)	1.169 (1.36 dB)	1.140 (1.14 dB)	1.118 (0.97 dB)
$M = 24$	1.133 (1.08 dB)	1.126 (1.03 dB)	1.099 (0.82 dB)	1.077 (0.64 dB)	1.064 (0.54 dB)	1.054 (0.45 dB)

For comparison, returning to the earlier scenario of a single stirring mechanism and single location ($M = 1$), Table 5 compares the ideal theoretical field nonuniformity levels between the case of *a priori* known distribution parameters (i.e., for $\mathcal{N} \rightarrow +\infty$) vs. sample-, i.e., test-based estimation of these parameters from the data set itself (i.e., for $\mathcal{N} = N$); cf. Secs. A.1.1 and A.1.3 for details. As expected, when $N \rightarrow +\infty$, both methods yield merging levels of field nonuniformity, whereas for small N the sample-based estimation gives rise to significantly larger nonuniformity levels⁵⁵. It is therefore recommended to estimate distribution parameters from a super-set of tuner or stir sweep data or from an independent method, whenever possible, in order to minimize the impact of uncertainty of distribution statistics on the estimated field nonuniformity and its uncertainty.

Table 5: Comparison of field nonuniformity coefficient $20 \log_{10}(1 + \sigma_{\text{Max}(|E_{R,\alpha}|)} / \langle \text{Max}(|E_{R,\alpha}|) \rangle)$ (in dB) at 95 % confidence level for *a priori* known ($\mathcal{N} \rightarrow +\infty$) vs. sample-based estimated ($\mathcal{N} = N$) distribution parameters, for a single stir mechanism and single measurement location ($M = 1$).

	$N = 1$	$N = 10$	$N = 100$	$N = 1\,000$	$N = 10\,000$
$\mathcal{N} \rightarrow +\infty$	3.65 dB	1.64 dB	0.98 dB	0.694 dB	0.5379 dB
$\mathcal{N} = N$	9.68 dB	1.92 dB	1.01 dB	0.697 dB	0.5382 dB

⁵⁵It is emphasized that the larger level in this latter case does of course not necessarily imply a larger physical field nonuniformity, but merely a larger *uncertainty* in estimating the field uniformity level. The latter is here calculated for an *ideal* random field and shows that the details of the estimation procedure can have a significant impact on the result for $N \sim 10$ or smaller. This uncertainty on the nonuniformity level may further increase for imperfect reverberation fields, i.e., nearer to the LUF. This requires further investigation, by using more appropriate parent pdfs.

5.3.6 Extensions

5.3.6.1 Sampling Distributions for Hybrid Stirring The previous analysis was based on the Gauss normal approximation for the ensemble distribution of the maximum power or field strength. For sample statistics related to $\chi_{2p}^{(2)}$ parent distributions, this strictly requires $N \rightarrow +\infty$, in order to apply the central limit theorem. In cases where N (i.e., the size of each data stir sequence from which one value of the sample statistic of interest is extracted) – but not necessarily M – is relatively large ($N > 30$), this is usually an excellent approximation. In some cases, however, one requires an estimate for the maximum power or field strength when N is not large. In this case, we must revert to the exact sampling distribution of the maximum value.

The starting point for finding the distribution of the sample maximum is the cdf of the ensemble maximum value for all possible sample sets of a given finite size N . This cdf, i.e.,

$$F_{\text{Max}(X)}[\max(X)] = [F_X(x = \max(X))]^N \quad (225)$$

is valid for $M \rightarrow +\infty$ and is now no longer approximated by a Gauss normal cdf, on account of the finiteness of N , but is calculable from the $\chi_{2p}^{(2)}$ parent distribution. The cdf (225) serves as the ensemble cdf for subsequent sampling M times. In this sampled distribution, the standard deviation is taken to be that of the *ensemble* distribution, which is an approximation because the standard deviation is itself a sample statistic, whose fluctuations are of the order of $1/\sqrt{M}$.

5.3.6.2 Sampling Distributions for Aggregated Sample Sets The previous analysis assumed the sample sets to remain grouped in sets of N sample points and extracting one sample maximum value per set. Another strategy consists in extracting just one sample maximum value from the single superset of all MN data points available. Under the condition that all these points have been evaluated under statistically equivalent conditions, the pdf for this maximum value follows as a straightforward extension from that for a single stir sequence, now with MN replacing N . Thus, for the sample average value, the distribution is given by (15) or (25) with $N \rightarrow MN$. For the maximum value⁵⁶,

$$f_{\text{Max}(X)}(\max(X); M, N) = MN[F_X(x = \max(X))]^{MN-1}f_X(x = \max(X)), \quad (226)$$

$$F_{\text{Max}(X)}(\max(X); M, N) = \left\{ [F_X(x = \max(X))]^N \right\}^M \rightarrow [F_X(x = \max(X))]^{MN} \quad (227)$$

⁵⁶Note that in (227), $\{[F_X(x = \max(X))]^N\}^M$ is *not* identically equal to $[F_X(x = \max(X))]^{MN}$, because in the former expression, the function between $\{\}$ represents a *sampling* cdf $G(\cdot; N)$ with respect to N . Rather, $\lim_{N \rightarrow +\infty} \{[F_X(x = \max(X))]^N\}^M = \lim_{N \rightarrow +\infty} [F_X(x = \max(X))]^{MN}$. Since we approximate large N by $N \rightarrow +\infty$ in our treatment, the equivalence of both forms can be assumed in this approximation.

in which $f_X(x)$ is the $\chi_{2p}^{(2)}$ parent distribution of X . Together with (11), the functions (226)–(227) are calculated with the aid of the underlying parent distributions, i.e.,

$$f_X(x) = \frac{p^{p/2}}{\Gamma(p) \sigma_X} \left(\frac{x}{\sigma_X} \right)^{p-1} \exp \left(-\sqrt{p} \frac{x}{\sigma_X} \right), \quad (228)$$

$$F_X(x) = \frac{\gamma \left(p, \sqrt{p} \frac{x}{\sigma_X} \right)}{\Gamma(p)} \quad (229)$$

for use with the maximum power, and

$$f_X(x) = \frac{2 \left[p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2 \right]^p}{\Gamma(p) \sigma_X} \left(\frac{x}{\sigma_X} \right)^{2p-1} \exp \left\{ - \left[p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2 \right] \left(\frac{x}{\sigma_X} \right)^2 \right\}, \quad (230)$$

$$F_X(x) = \frac{\gamma \left(p, \left[p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2 \right] \left(\frac{x}{\sigma_X} \right)^2 \right)}{\Gamma(p)} \quad (231)$$

for use with the maximum field strength.

5.3.6.3 Exact Sampling Distributions for Small Numbers of Independent Stir States

For relatively small value of N , the small-sample effects of both M and N must be taken into account. In this case, we must take recourse to the sampling distribution of P_R with unknown estimated distribution parameters. Details of the derivation of this pdf are given in Appendix A. The result is a scaled Fisher–Snedecor F parent pdf for the power, given by (268), i.e.,

$$f_{P_R}(p_R; \mathcal{N}) \simeq \frac{\Gamma(p\mathcal{N} - \frac{1}{2} + p)}{\Gamma(p\mathcal{N} - \frac{1}{2})\Gamma(p)} \frac{p^{p/2}}{\left(p\mathcal{N} - \frac{1}{2}\right)^p s_{P_R}} \frac{\left(\frac{p_R}{s_{P_R}}\right)^{p-1}}{\left(1 + \frac{\sqrt{p}}{p\mathcal{N} - \frac{1}{2}} \frac{p_R}{s_{P_R}}\right)^{p\mathcal{N} - \frac{1}{2} + p}} \quad (232)$$

with $p = 1$ or 3 for Cartesian or vectorial power, respectively, and $\mathcal{N}$ representing the number of degrees of freedom of the parent sampling distribution on the basis of which the distribution parameter s_{P_R} is estimated. This parameter should not be confused with N , which is the number of independent samples that are (or can be) generated during the immunity test. Typically, $\mathcal{N} \geq N$ in order to get a sufficiently accurate parent distribution and estimates. In case of stirring combined with spatial sampling at multiple locations, a good choice is $\mathcal{N} = MN$. For $\mathcal{N} \rightarrow +\infty$, we retrieve of course a parent distribution with known, i.e., deterministic parameter values.

For the field strength, the sampling parent pdf follows from nonlinear transformation of (268) and is given approximately by (278), i.e.,

$$f_{|E_R|}(|e_R|; \mathcal{N}) \simeq \frac{\Gamma(p\mathcal{N} - \frac{1}{2} + p)}{\Gamma(p\mathcal{N} - \frac{1}{2})\Gamma(p)} \frac{2}{\left(p\mathcal{N} - \frac{1}{2}\right)^p s_{|E_R|}} \left[p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2 \right]^p \frac{\left(\frac{|e_R|}{s_{|E_R|}} \right)^{2p-1}}{\left[1 + \frac{p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2}{p\mathcal{N} - \frac{1}{2}} \left(\frac{|e_R|}{s_{|E_R|}} \right)^2 \right]^{p\mathcal{N} - \frac{1}{2} + p}}. \quad (233)$$

In Figures 33–36, $\eta\%$ confidence intervals of the sample maximum based on this approximation are compared with the exact intervals for these sampling distributions. For the latter, the number of degrees of freedom was taken to be equal to the number of sample points, i.e., the distribution parameters are estimated from the test data themselves in these plots ($\mathcal{N} = N$, $M = 1$). These Figures are to be compared to the corresponding plots in Figures 25–26 and 27–28 that were based on ensemble values of the distribution parameters. It can be witnessed from Figures 33–36 that for the sampling distributions (232) and (233), the estimate of the confidence interval obtained from the expanded uncertainty of the Gauss normal distribution is now a much poorer approximation.

With (232) and (233), the maximum-value pdfs can be obtained in the usual way as

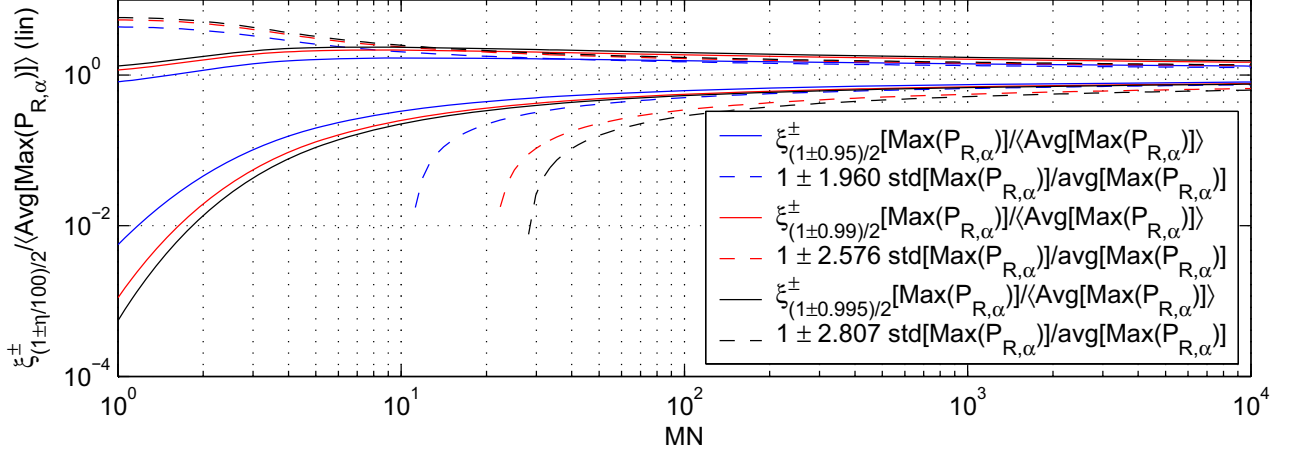
$$f_{\text{Max}(P_R)}[\max(P_R); N] = N f_{P_R}[p_R = \max(P_R); N] [F_{P_R}[p_R = \max(P_R); N]]^{N-1} \quad (234)$$

$$f_{\text{Max}(|E_R|)}[\max(|E_R|); N] = N f_{|E_R|}[|e_R| = \max(|E_R|); N] \left[F_{|E_R|}[|e_R| = \max(|E_R|); N] \right]^{N-1} \quad (235)$$

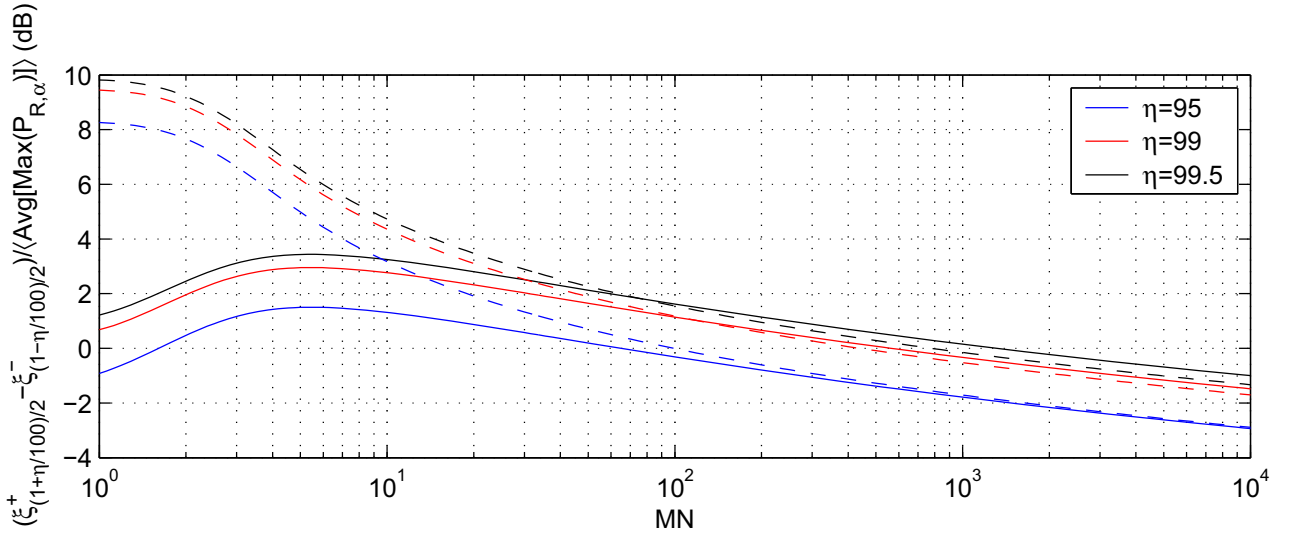
Despite employing the sampling pdfs of P_R or $|E_R|$, the resulting pdfs for $\text{Max}(P_R)$ and $\text{Max}(|E_R|)$ are still approximations, in the sense that $s_{P_R} f_{P_R}(p_R; \mathcal{N})$ and $s_{|E_R|} f_{|E_R|}(|e_R|; \mathcal{N})$ are the (exact) pdfs of $f_{P_R/s_{P_R}}(p_R/s_{P_R}; \mathcal{N})$ and $f_{|E_R|/s_{|E_R|}}(|e_R|/s_{|E_R|}; \mathcal{N})$, respectively, i.e., for the standardized power and field strength rather than for the power and field strength themselves. The pdfs (234)–(235) with the expressions in (232) and (233) are accurate approximations of $s f_{P_R}(p_R; \mathcal{N})$ and $f_{|E_R|}(|e_R|; \mathcal{N})$ insofar as s_{P_R} and $s_{|E_R|}$ are accurate estimates of σ_{P_R} and $\sigma_{|E_R|}$, i.e., for sufficiently large values of N . More generally, we will show in Part II that exact sampling pdfs for arbitrary including small $\mathcal{N} = N$ (i.e., for sample maximum and sample statistics both being extracted from the same sample of size N) are given in integral form as

$$\begin{aligned} f_{\text{Max}(P_R)}[\max(P_R); N] &= \frac{p^{p/2} N (pN - \frac{1}{2})^{pN - \frac{1}{2}}}{[\Gamma(p)]^N \Gamma(pN - \frac{1}{2}) \sigma_{P_R}^{pN - \frac{1}{2}}} [\max(P_R)]^{p-1} \\ &\times \int_0^{+\infty} x^{p(N-1) - \frac{3}{2}} \exp \left[- \left(pN - \frac{1}{2} \right) \frac{x}{\sigma_{P_R}} - \sqrt{p} \frac{\max(P_R)}{x} \right] \left[\gamma \left(p, \sqrt{p} \frac{\max(P_R)}{x} \right) \right]^{N-1} dx, \quad (236) \\ f_{\text{Max}(|E_R|)}[\max(|E_R|); N] &= \frac{4N \left[p - \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \right)^2 \right]^p (pN - \frac{1}{2})^{pN - \frac{1}{2}}}{[\Gamma(p)]^N \Gamma(pN - \frac{1}{2}) \sigma_{|E_R|}^{2pN-1}} [\max(|E_R|)]^{2p-1} \\ &\times \int_0^{+\infty} x^{2[p(N-1)-1]} \exp \left\{ - \left(pN - \frac{1}{2} \right) \left(\frac{x}{\sigma_{|E_R|}} \right)^2 - \left[p - \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \right)^2 \right] \left(\frac{\max(|E_R|)}{x} \right)^2 \right\} \\ &\times \left[\gamma \left(p, \left[p - \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \right)^2 \right] \left(\frac{\max(|E_R|)}{x} \right)^2 \right) \right]^{N-1} dx. \quad (237) \end{aligned}$$

These forms can be used for determining the expected value of the sample maximum or sample maximum-to-average value and associated $\eta\%$ confidence intervals when N is relatively small so that correlation between $\text{Max}(\cdot)$ and $\text{Avg}(\cdot)$ cannot be neglected.



(a)



(b)

Figure 33: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\langle \text{Max}(P_{R,\alpha}) \rangle$ ($p = 1$): (solid lines) exact, based on percentiles of maximum-value distribution of $P_{R,\alpha}$ using sample-based estimates of distribution parameters ($F_{2,2N-1}$ parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Max}(P_{R,\alpha})}/\text{avg}(\text{Max}(P_{R,\alpha})), 1 + 1.960 s_{\text{Max}(P_{R,\alpha})}/\text{avg}(\text{Max}(P_{R,\alpha}))]$, $[1 - 2.576 s_{\text{Max}(P_{R,\alpha})}/\text{avg}(\text{Max}(P_{R,\alpha})), 1 + 2.576 s_{\text{Max}(P_{R,\alpha})}/\text{avg}(\text{Max}(P_{R,\alpha}))]$, and $[1 - 2.807 s_{\text{Max}(P_{R,\alpha})}/\text{avg}(\text{Max}(P_{R,\alpha})), 1 + 2.807 s_{\text{Max}(P_{R,\alpha})}/\text{avg}(\text{Max}(P_{R,\alpha}))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.

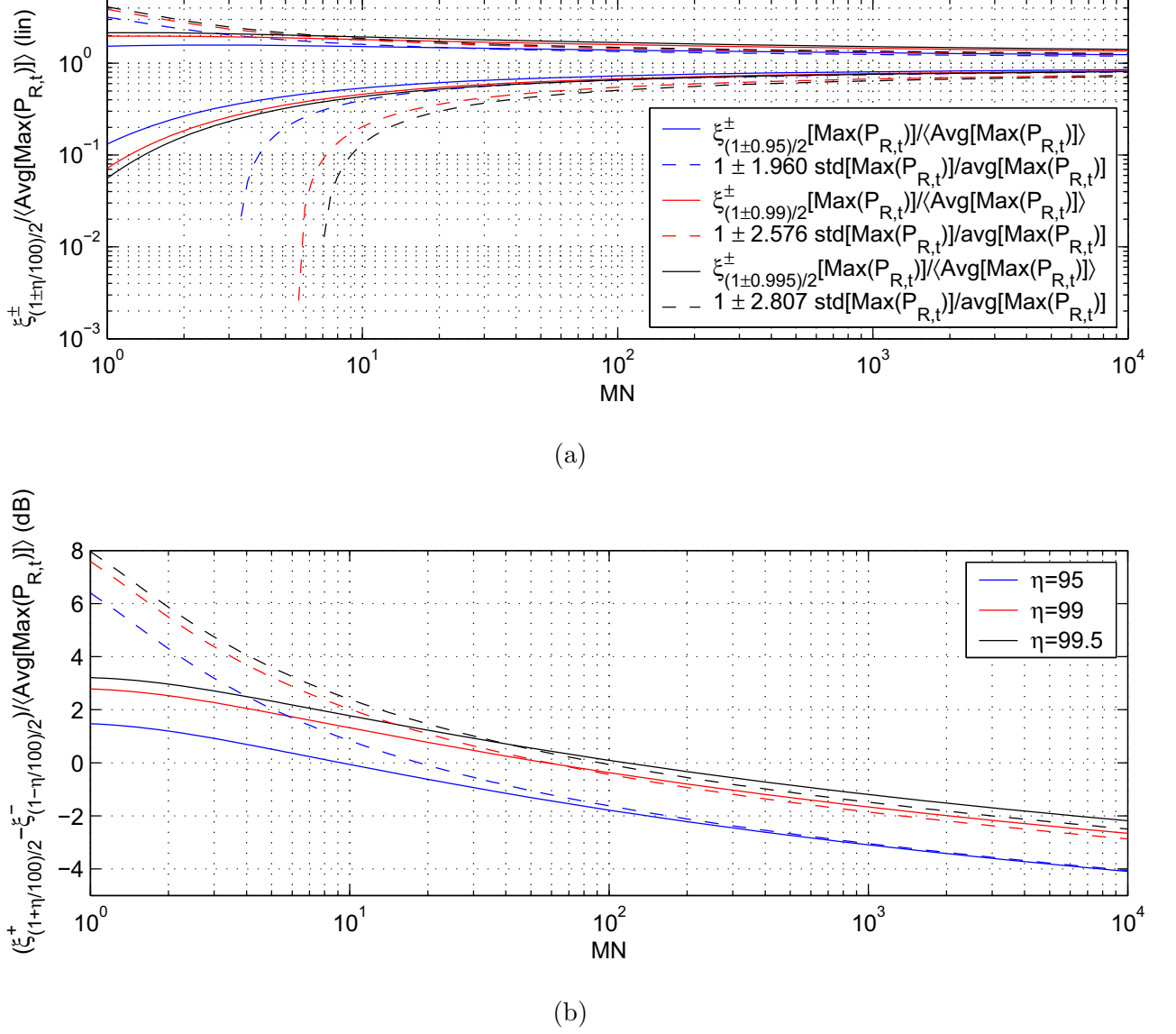


Figure 34: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\langle \text{Max}(P_{R,t}) \rangle$ ($p = 3$): (solid lines) exact, based on percentiles of maximum-value distribution of $P_{R,t}$ using sample-based estimates of distribution parameters ($F_{6,6N-1}$ parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha})), 1 + 1.960 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha}))]$, $[1 - 2.576 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha})), 1 + 2.576 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha}))]$, and $[1 - 2.807 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha})), 1 + 2.807 s_{\text{Max}(P_{R,\alpha})} / \text{avg}(\text{Max}(P_{R,\alpha}))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.

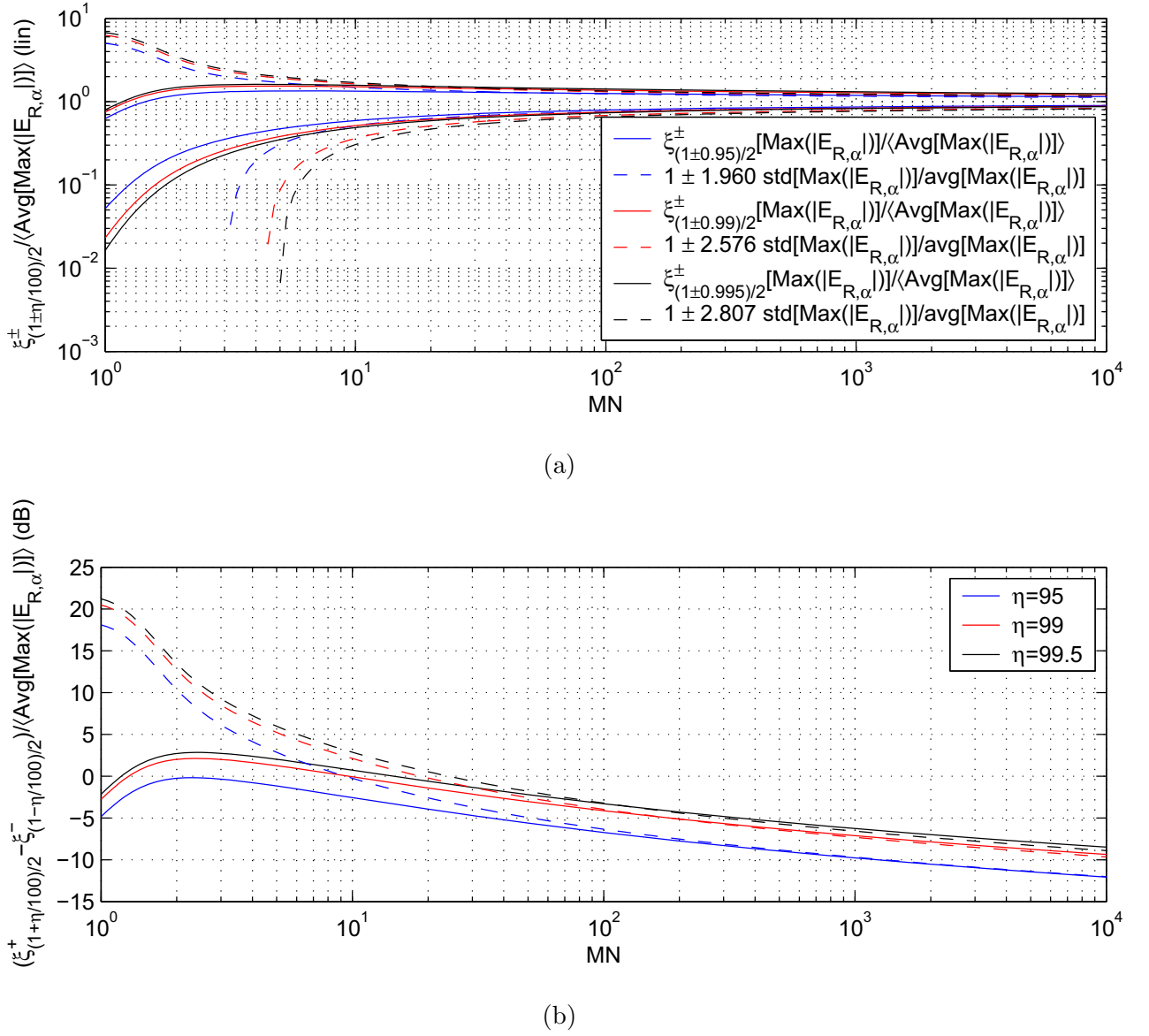
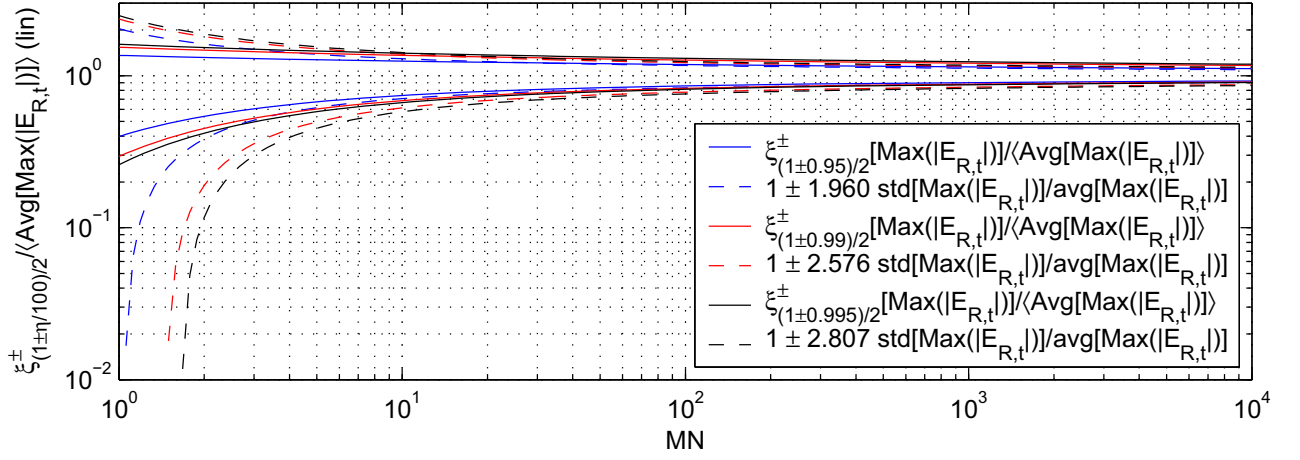
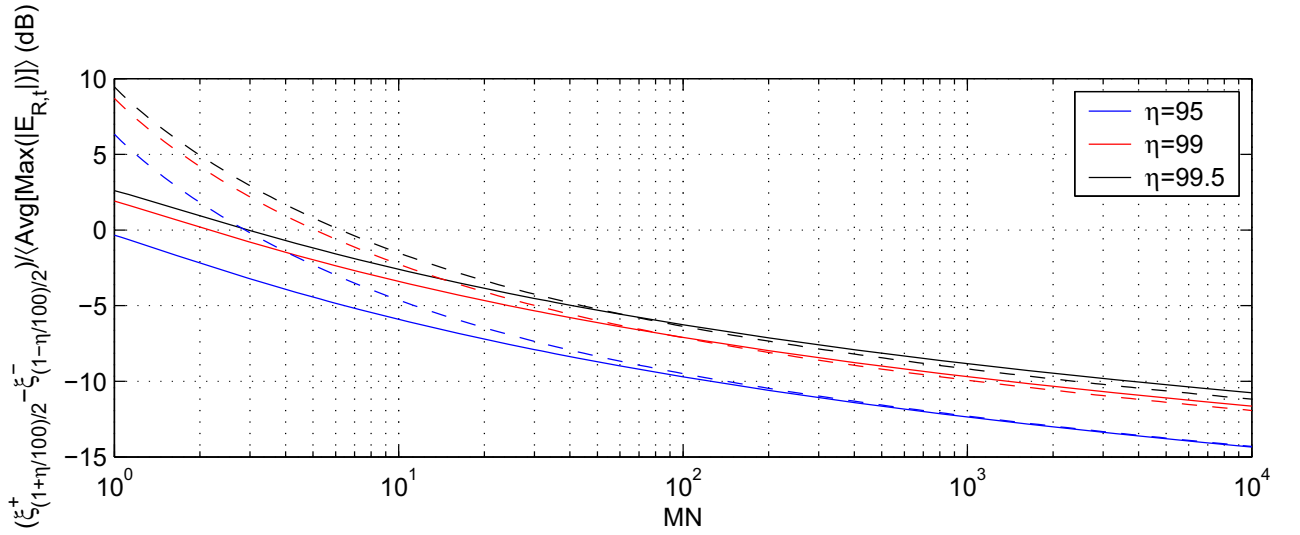


Figure 35: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\langle \text{Max}(|E_{R,\alpha}|) \rangle$ ($p = 1$): (solid lines) exact, based on percentiles of maximum-value distribution of $|E_{R,\alpha}|$ using sample-based estimates of distribution parameters (transformed $F_{2,2N-1}$ parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|)), 1 + 1.960 s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|))]$, $[1 - 2.576 s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|))], 1 + 2.576 s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|))]$, and $[1 - 2.807 s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|))], 1 + 2.807 s_{\text{Max}(|E_{R,\alpha}|)}/\text{avg}(\text{Max}(|E_{R,\alpha}|))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.



(a)



(b)

Figure 36: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\langle \text{Max}(|E_{R,t}|) \rangle$ ($p = 3$): (solid lines) exact, based on percentiles of maximum-value distribution of $|E_{R,t}|$ using sample-based estimates of distribution parameters (transformed $F_{6,6N-1}$ parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Max}(|E_{R,t}|)} / \text{avg}(\text{Max}(|E_{R,t}|)), 1 + 1.960 s_{\text{Max}(|E_{R,t}|)} / \text{avg}(\text{Max}(|E_{R,t}|))]$, $[1 - 2.576 s_{\text{Max}(|E_{R,t}|)} / \text{avg}(\text{Max}(|E_{R,t}|)), 1 + 2.576 s_{\text{Max}(|E_{R,t}|)} / \text{avg}(\text{Max}(|E_{R,t}|))]$, and $[1 - 2.807 s_{\text{Max}(|E_{R,t}|)} / \text{avg}(\text{Max}(|E_{R,t}|)), 1 + 2.807 s_{\text{Max}(|E_{R,t}|)} / \text{avg}(\text{Max}(|E_{R,t}|))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.

5.4 Experimental Results

The theoretical results were compared with measurements inside the NPL rectangular reverberation chamber for two frequency ranges: the first one from 200 to 1500 MHz, using a pair of log-periodic antennas, and the second one from 1 GHz to 18 GHz, using a pair of dual-ridge horn antennas.

In order to compare theoretical and experimental data, the measured data is normalized by the thru measured data (i.e., with and end-to-end cable interconnection after detaching antennas) and the effective aperture of the antennas. This eliminates the dependence on $\langle P_T \rangle$. The statistics of the measured values of the received field strength are obtained from

$$\text{avg}(|E_{R,\alpha}|) \propto \sqrt{\frac{8\pi\eta_0}{3\lambda^2} \frac{\text{avg}(|S_{21}|^2)}{|S_{21,\text{cable}}|^2}} \quad (238)$$

$$\text{std}(|E_{R,\alpha}|) \propto \sqrt{\frac{4-\pi}{\pi} \frac{8\pi\eta_0}{3\lambda^2} \frac{\text{std}(|S_{21}|^2)}{|S_{21,\text{cable}}|^2}} \quad (239)$$

$$\text{avg}(\text{Max}(|E_{R,\alpha}|)) = \text{avg}(|E_{R,\alpha}|) \sqrt{\frac{4}{\pi} \left(0.5772 + \ln(N+1) - \frac{1}{2(N+1)} \right)} \quad (240)$$

$$\text{std}(\text{Max}(|E_{R,\alpha}|)) = \text{avg}(|E_{R,\alpha}|) \sqrt{\frac{1}{4-\pi} \left(\frac{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}{0.5772 + \ln(N+1) - \frac{1}{2(N+1)}} \right)} \quad (241)$$

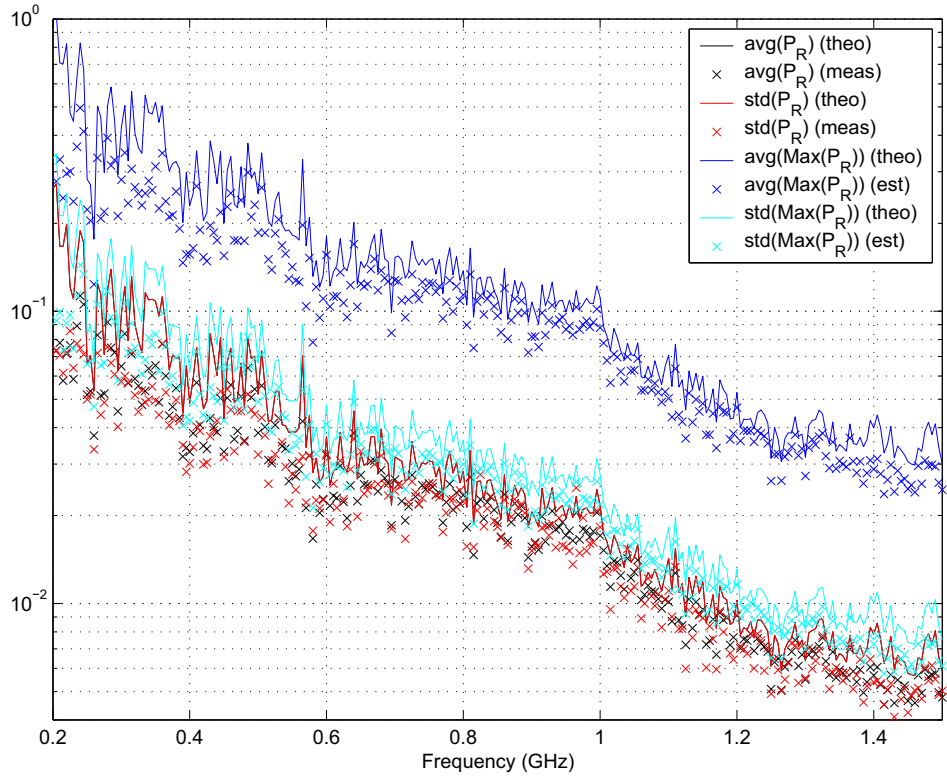
whereas for the measured received power,

$$\text{avg}(P_{R,\alpha}) \propto \frac{\text{avg}(|S_{21}|^2)}{|S_{21,\text{cable}}|^2} \quad (242)$$

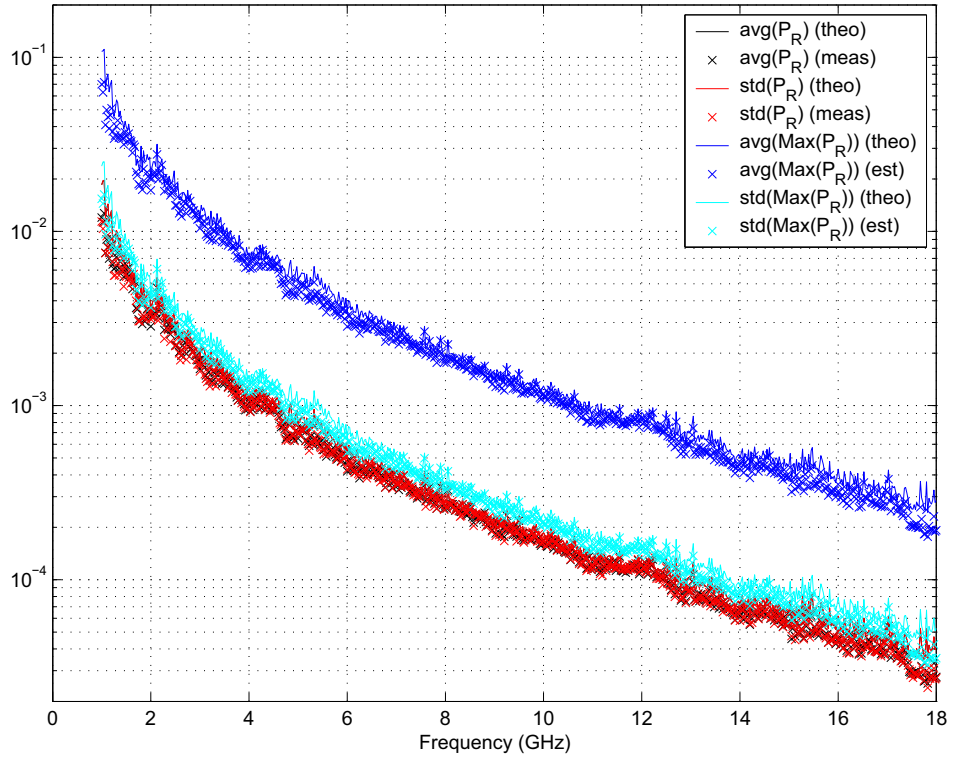
$$\text{std}(P_{R,\alpha}) \propto \frac{\text{std}(|S_{21}|^2)}{|S_{21,\text{cable}}|^2} \quad (243)$$

$$\text{avg}(\text{Max}(P_{R,\alpha})) = \text{avg}(P_{R,\alpha}) \left(0.5772 + \ln(N+1) - \frac{1}{2(N+1)} \right) \quad (244)$$

$$\text{std}(\text{Max}(P_{R,\alpha})) = \text{std}(P_{R,\alpha}) \sqrt{\frac{\pi^2}{6} - \frac{N+1}{N(N+2)}}. \quad (245)$$

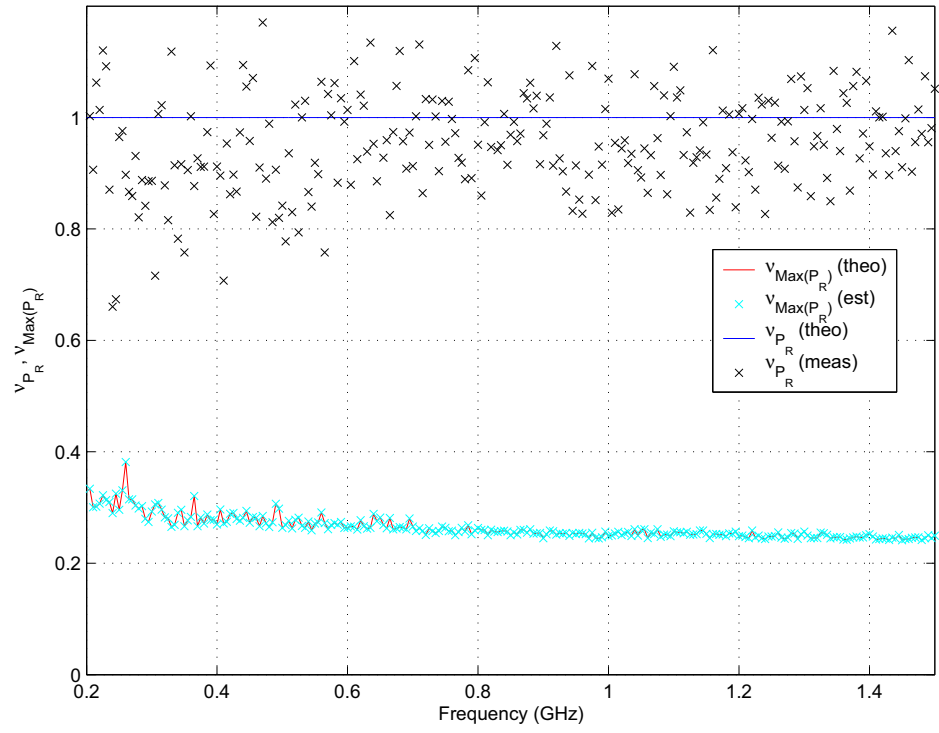


(a)

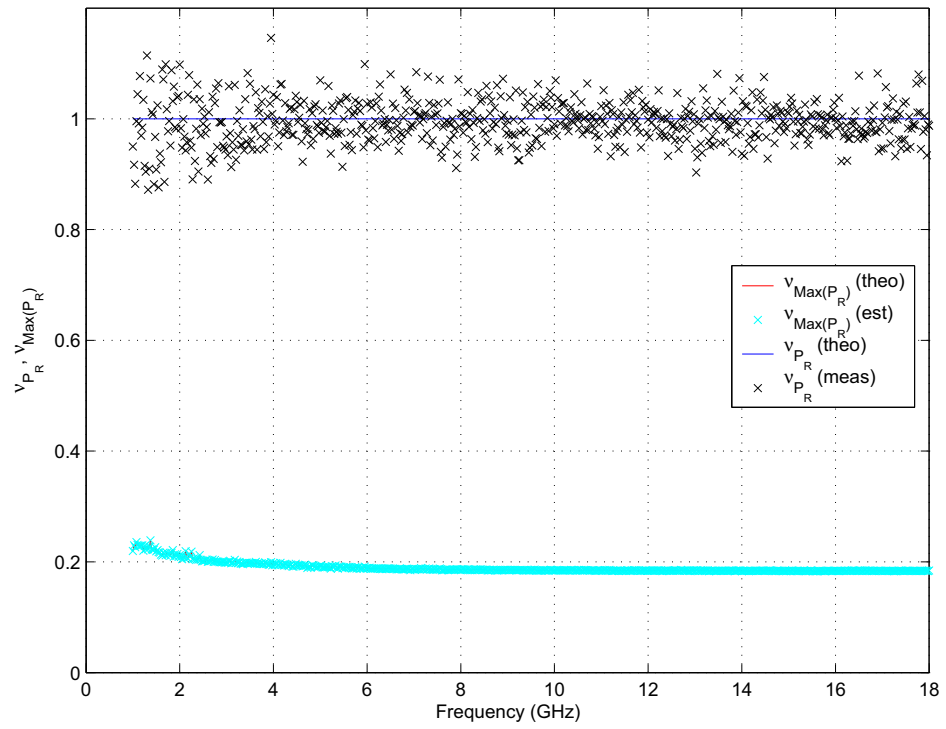


(b)

Figure 37: Average and standard deviation of $P_{R,\alpha}$ and $\text{Max}(P_{R,\alpha})$ (up to a normalization constant): comparison between theoretical (theo) and measured (meas) or estimated (est) values.

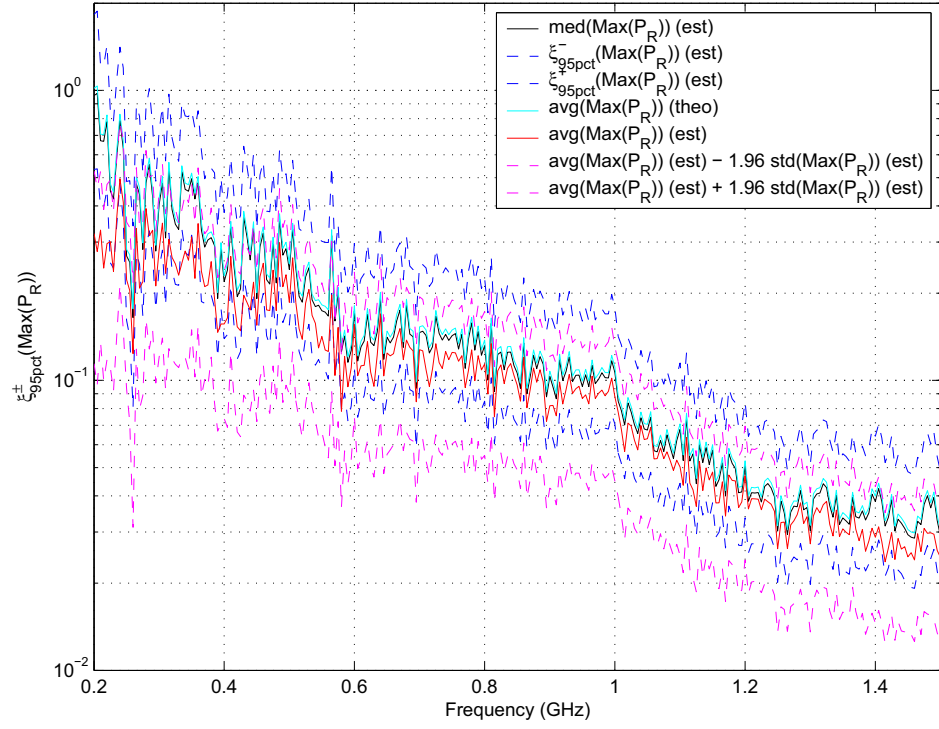


(a)

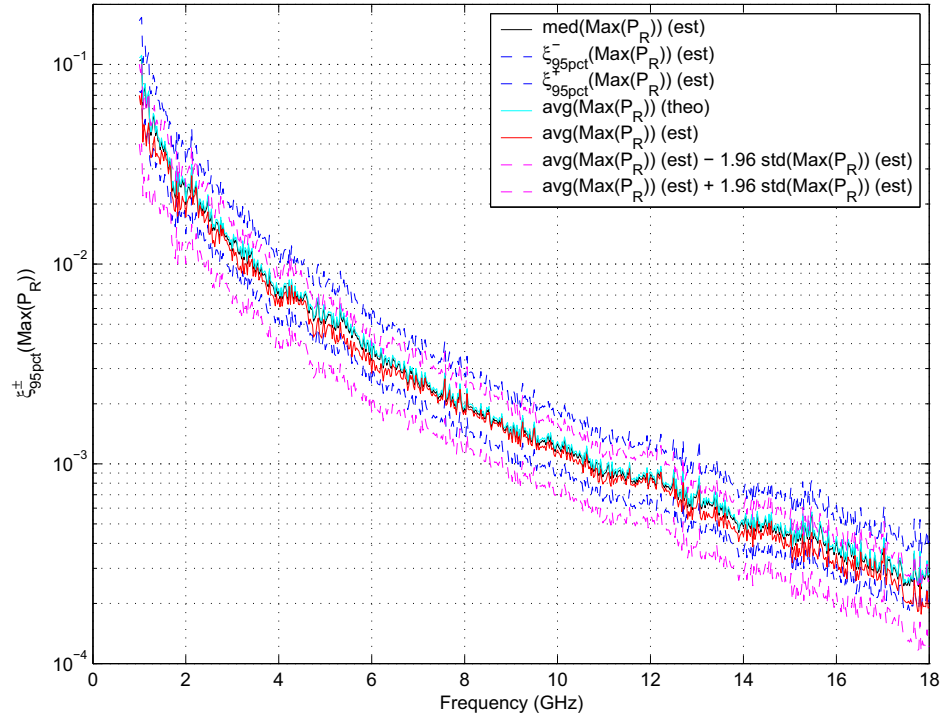


(b)

Figure 38: Coefficient of variation for $P_{R,\alpha}$ and $\text{Max}(P_{R,\alpha})$: comparison between theoretical and measured or estimated values.

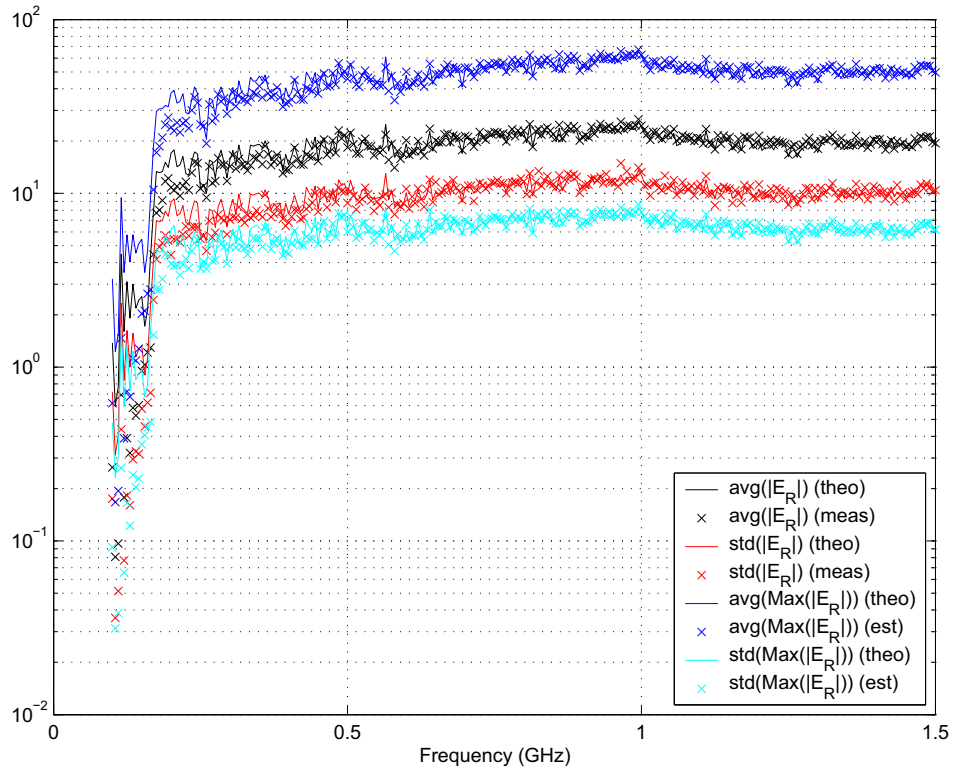


(a)

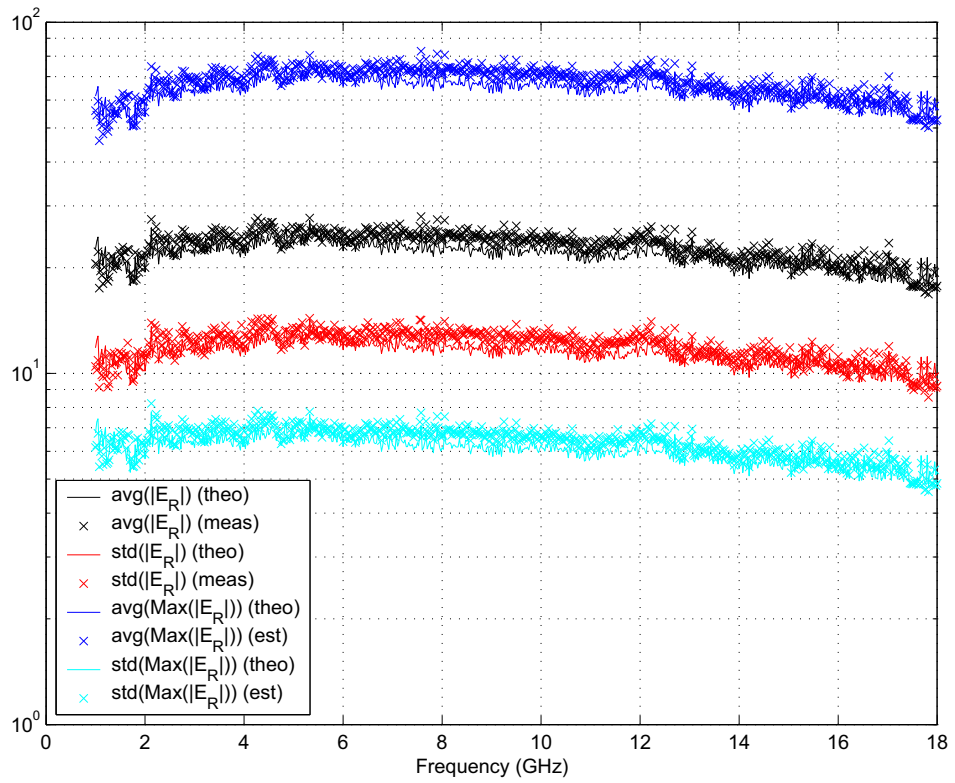


(b)

Figure 39: Average, median, and 95 % confidence interval for $\text{Max}(P_{R,\alpha})$: comparison between theoretical and estimated values. The interval $\text{avg}(\text{Max}(P_{R,\alpha})) (\text{est}) - 1.960 \text{std}(\text{Max}(P_{R,\alpha})) (\text{est}) \leq \text{avg}(\text{Max}(P_{R,\alpha})) \leq \text{avg}(\text{Max}(P_{R,\alpha})) (\text{est}) + 1.960 \text{std}(\text{Max}(P_{R,\alpha})) (\text{est})$, strictly valid for $N \rightarrow +\infty$, is shown for reference only.

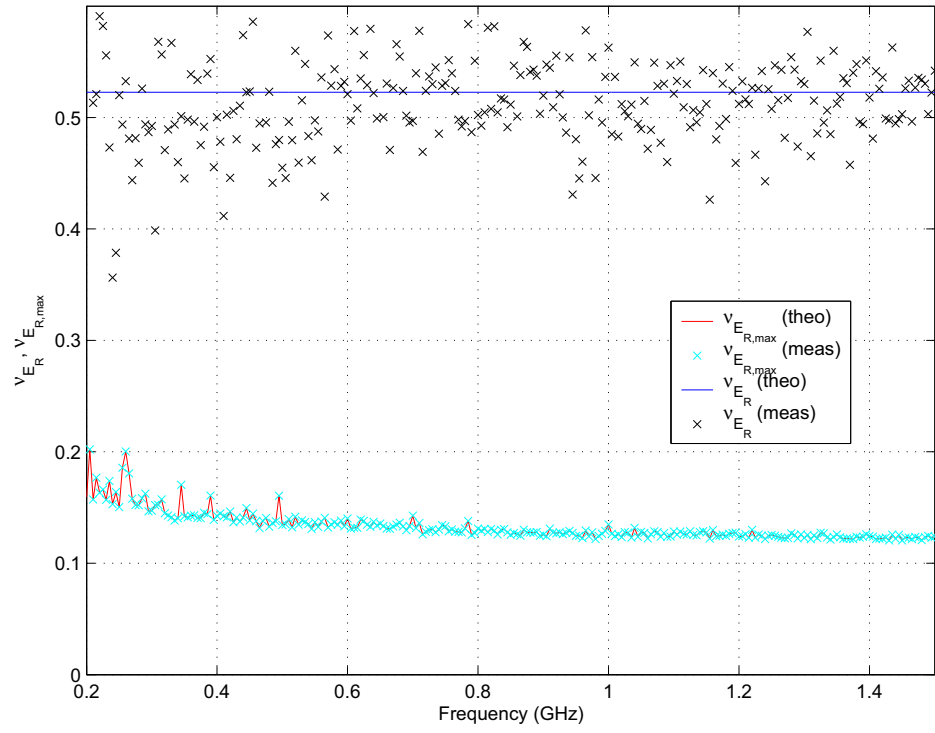


(a)

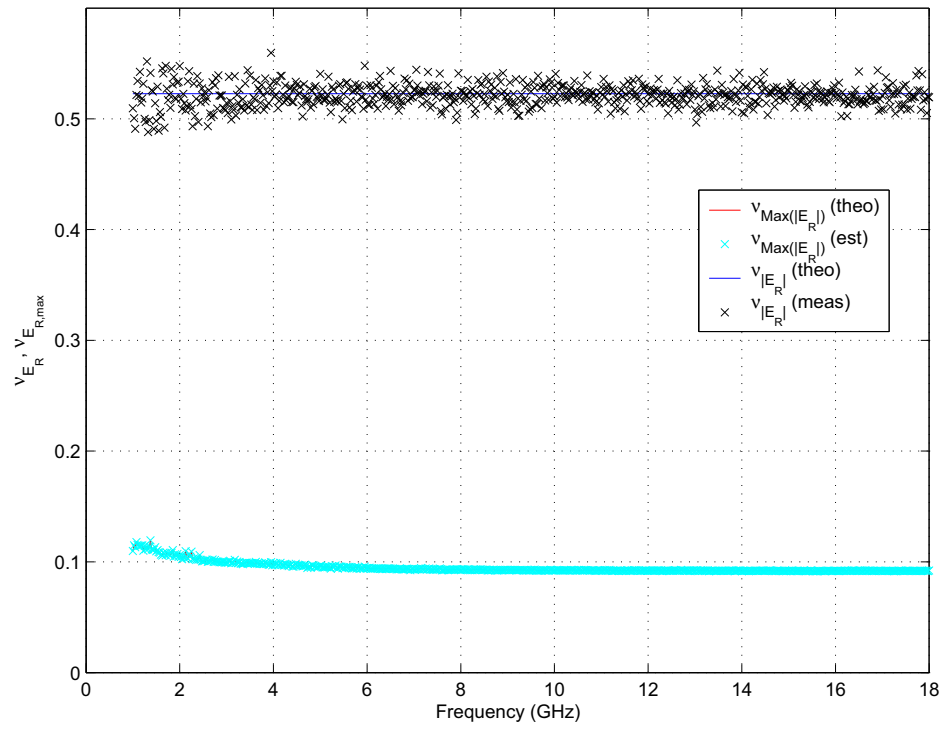


(b)

Figure 40: Average and standard deviation of $|E_{R,\alpha}|$ and $\text{Max}(|E_{R,\alpha}|)$ (up to a normalization constant): comparison between theoretical and measured or estimated values.

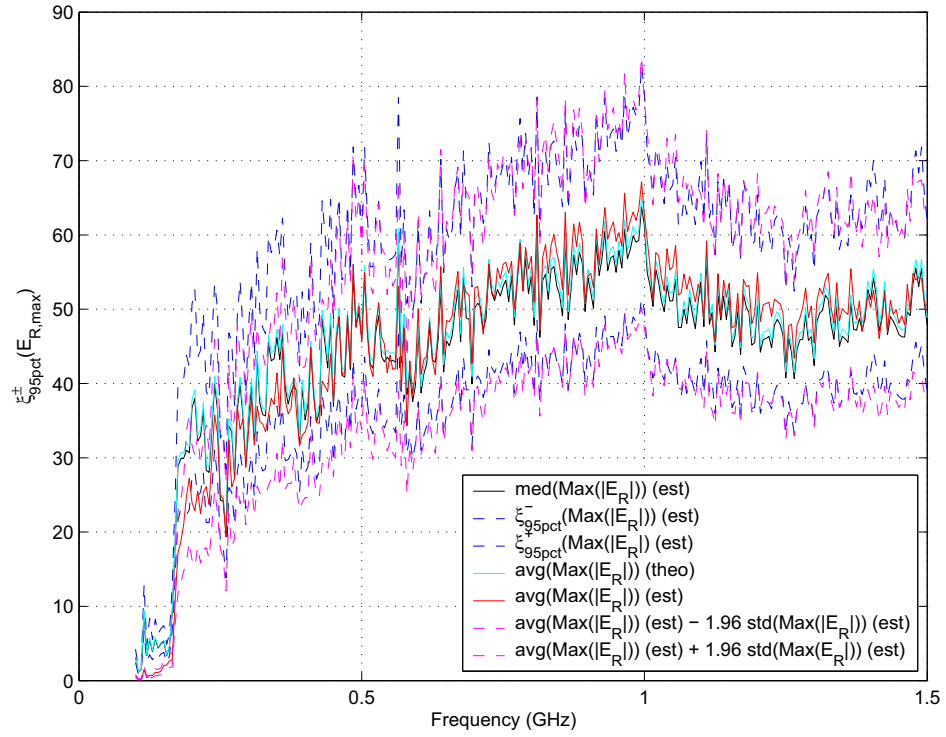


(a)

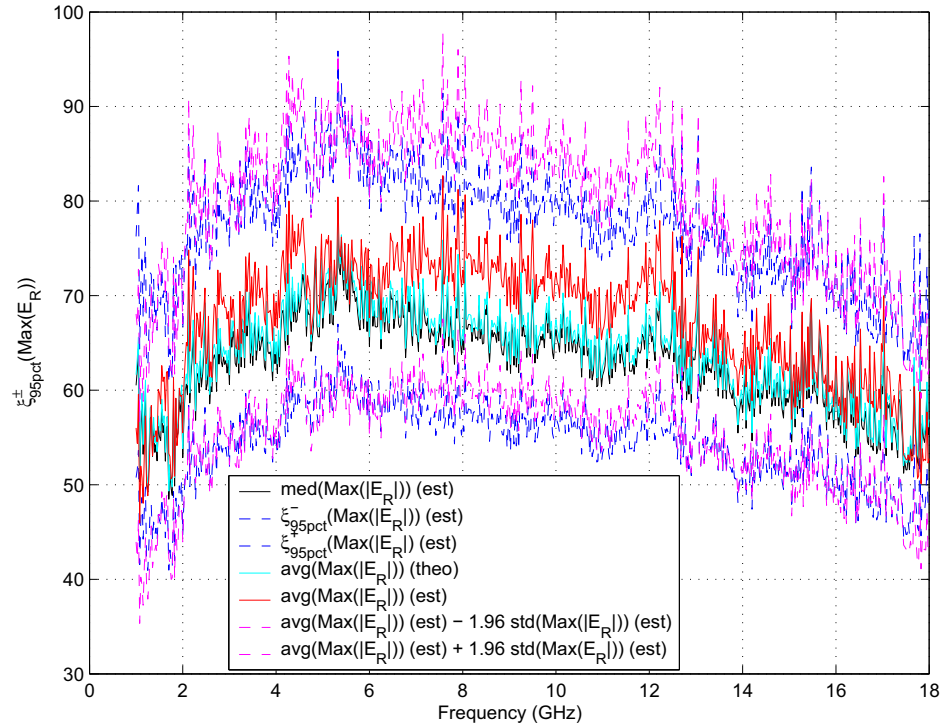


(b)

Figure 41: Coefficient of variation for $|E_{R,\alpha}|$ and $\text{Max}(|E_{R,\alpha}|)$: comparison between theoretical and measured values.



(a)



(b)

Figure 42: Average, median, and 95 % confidence interval for $|E_{R,\alpha}|$ and $\text{Max}(|E_{R,\alpha}|)$: comparison between theoretical and measured or estimated values. The interval $\text{avg}(\text{Max}(|E_{R,\alpha}|)) (\text{est}) - 1.960 \text{std}(\text{Max}(|E_{R,\alpha}|)) (\text{est}) \leq \text{avg}(\text{Max}(|E_{R,\alpha}|)) (\text{est}) \leq \text{avg}(\text{Max}(|E_{R,\alpha}|)) (\text{est}) + 1.960 \text{std}(\text{Max}(|E_{R,\alpha}|)) (\text{est})$ is shown for reference only.

6 Susceptibility

First, we clarify the terminology adopted here. The notion of susceptibility is often used in EMC as a synonym for immunity or vulnerability with respect to *high* EM field strength. Here, instead, we use the term susceptibility in the meaning of responsiveness or sensitivity of an EUT to relatively *small* field strengths or intensities, i.e., as a measure for sensitivity⁵⁷ to weak fields.

For estimates of the ensemble minimum value of a multi-dimensional stirring process based on hierarchical 2D sampling (i.e., by considering M stir sequences one by one, followed by subsampling), the pdf and cdf are *not* obtained by mere substitution of MN for N in the distribution for the single-sequence minimum, as a consequence of the different functional form of $F_{\text{Min}(X)}$ compared to $F_{\text{Max}(X)}$. Generally, for 1-D sampling,

$$F_{\text{Min}(X)}(\min(X); N) = 1 - [1 - F_X(x = \min(X))]^N. \quad (246)$$

Therefore, for hierarchical 2D sampling

$$\begin{aligned} f_{\text{Min}(X)}(\min(X); M, N) &= MN \left\{ 1 - [1 - (1 - F_X(x = \min(X)))^N] \right\}^{M-1} \\ &\quad \times [1 - F_X(x = \min(X))]^{N-1} f_X(x = \min(X)) \end{aligned} \quad (247)$$

$$F_{\text{Min}(X)}(\min(X); M, N) = 1 - \left\{ 1 - [1 - (1 - F_X(x = \min(X)))^N] \right\}^M. \quad (248)$$

In (248), the bracketed term corresponds to the marginal cdf of the minimum value with respect to N -sized sample sets, which constitutes the parent cdf for subsequent sampling across the M minima.

On the other hand, when all sample values are amalgamated simultaneously into a single superset of MN samples before identifying the overall minimum value, the distributions and statistics of the associated (single) sample minimum value are indeed

$$f_{\text{Min}(X)}(\min(X); MN) = MN [1 - F_X(x = \min(X))]^{MN-1} f_X(x = \min(X)) \quad (249)$$

$$F_{\text{Min}(X)}(\min(X); MN) = 1 - [1 - F_X(x = \min(X))]^{MN}. \quad (250)$$

For the latter case, the confidence interval boundaries and widths, together with their values for Gauss normal approximations of the minimum-value distributions, are shown in Figs. 43–46. The generally weak dependence of the limits and width of the confidence intervals on MN is immediately apparent.

⁵⁷Sensitivity generally refers to the ability to register small changes of an EN quantity, whence sensitivity is with reference to $|E_R| = 0$.

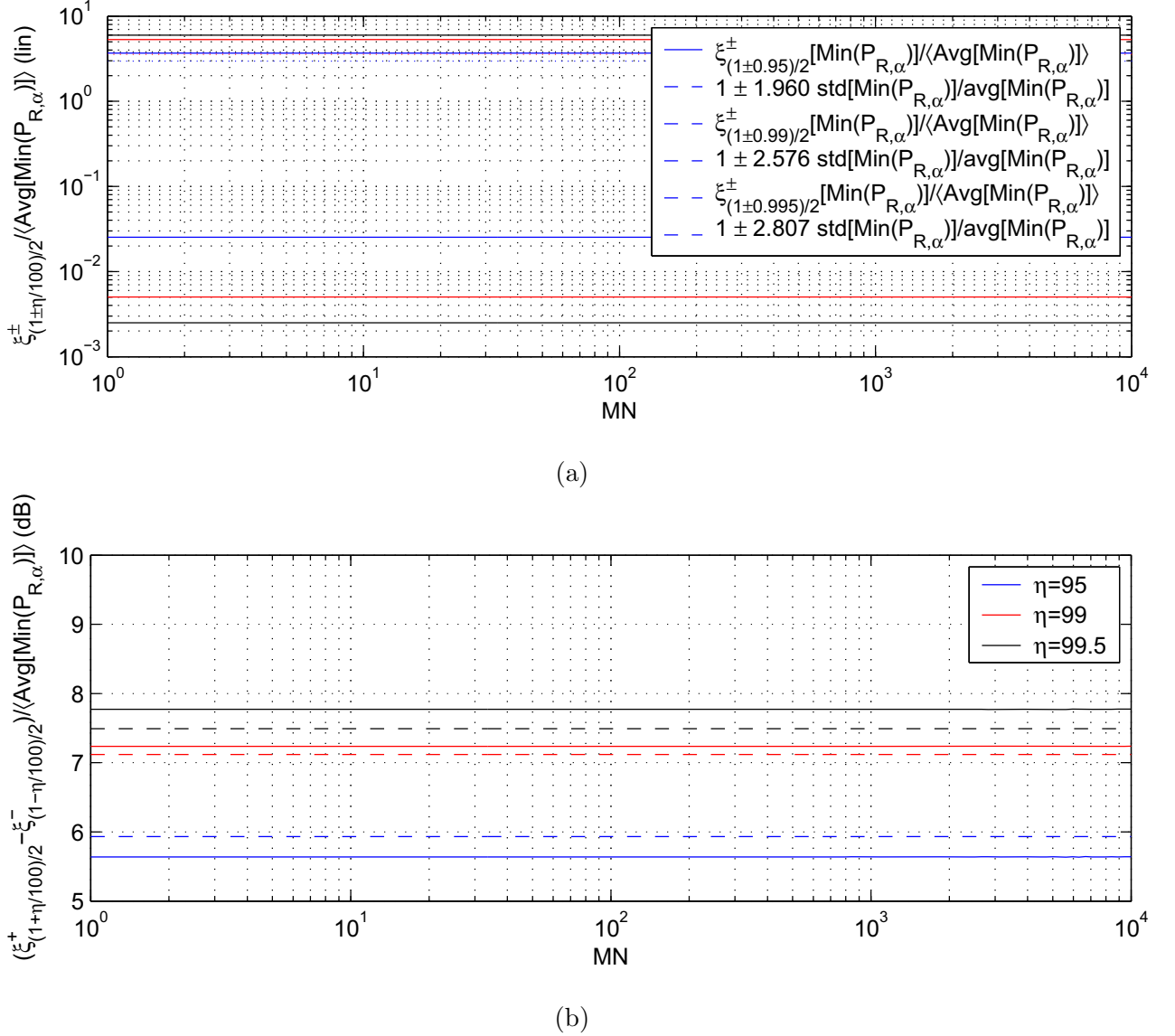


Figure 43: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Min}(P_{R,\alpha})$ ($p = 1$): (solid lines) exact, based on percentiles of minimum-value distribution of $P_{R,\alpha}$ (χ^2_{2N} parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Min}(P_{R,\alpha})} / \text{avg}(\text{Min}(P_{R,\alpha})), 1 + 1.960 s_{\text{Min}(P_{R,\alpha})} / \text{avg}(\text{Min}(P_{R,\alpha}))]$, $[1 - 2.576 s_{\text{Min}(P_{R,\alpha})} / \text{avg}(\text{Min}(P_{R,\alpha})), 1 + 2.576 s_{\text{Min}(P_{R,\alpha})} / \text{avg}(\text{Min}(P_{R,\alpha}))]$, and $[1 - 2.807 s_{\text{Min}(P_{R,\alpha})} / \text{avg}(\text{Min}(P_{R,\alpha})), 1 + 2.807 s_{\text{Min}(P_{R,\alpha})} / \text{avg}(\text{Min}(P_{R,\alpha}))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.

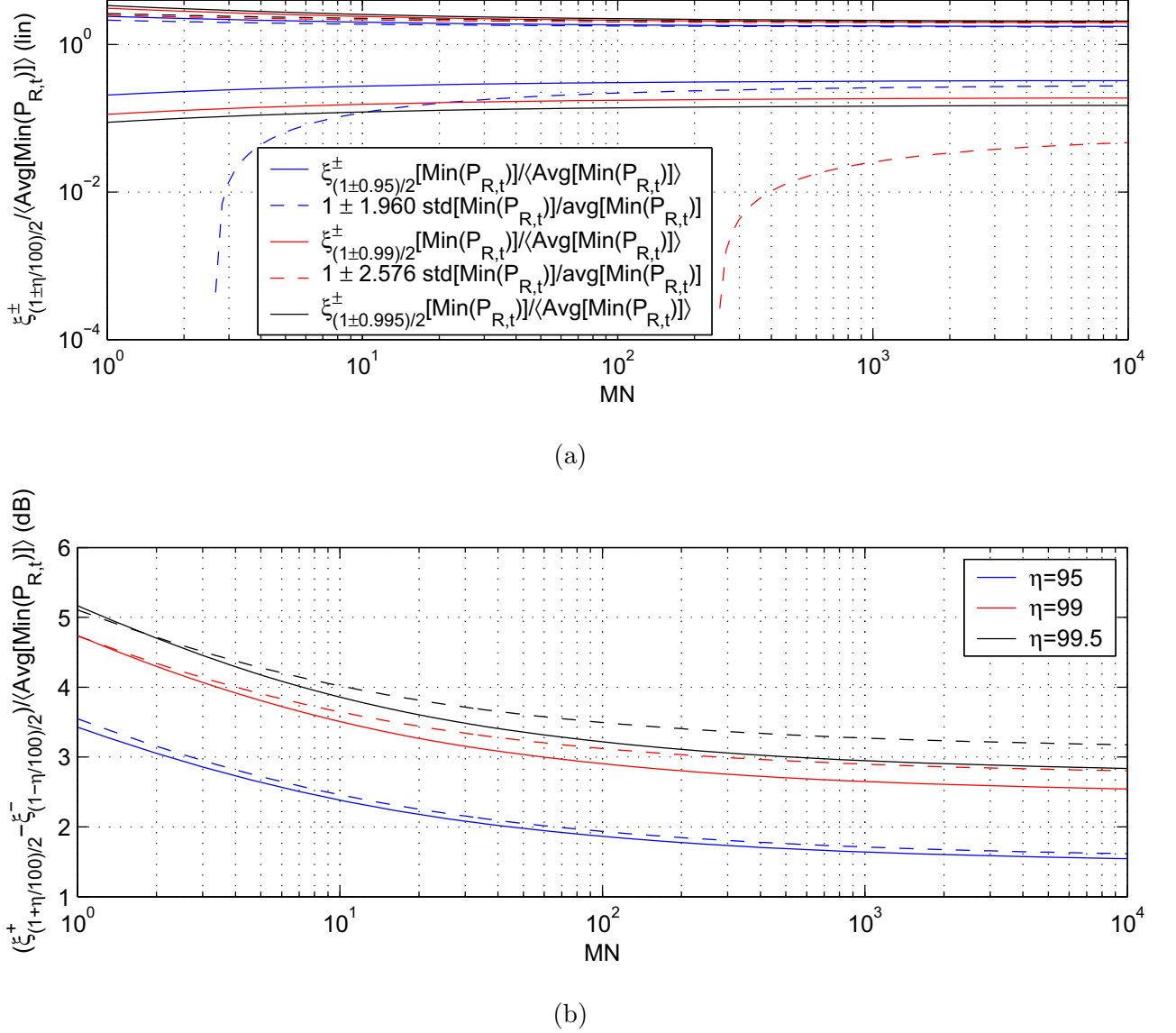
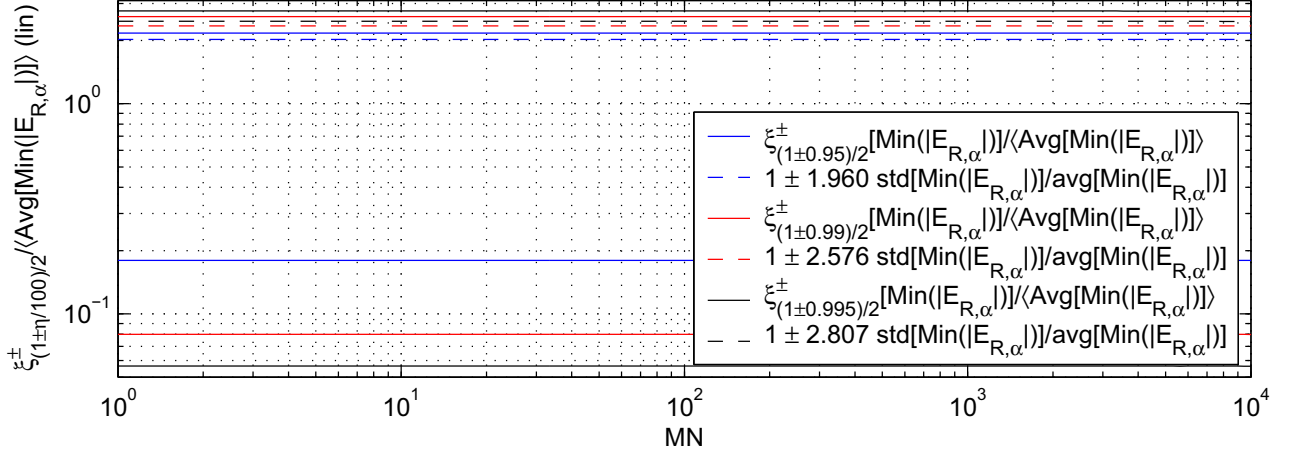
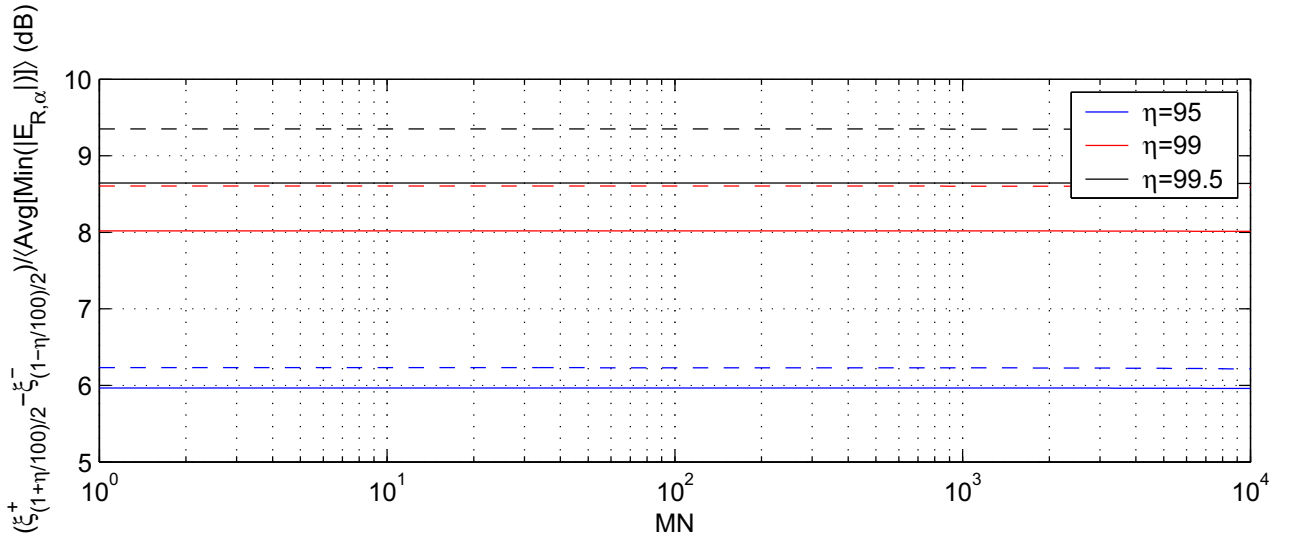


Figure 44: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Min}(P_{R,t})$ ($p = 3$): (solid lines) exact, based on percentiles of minimum-value distribution of $P_{R,t}$ (χ_{6N}^2 parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 s_{\text{Min}(P_{R,t})}/\text{avg}(\text{Min}(P_{R,t})), 1 + 1.960 s_{\text{Min}(P_{R,t})}/\text{avg}(\text{Min}(P_{R,t}))]$, $[1 - 2.576 s_{\text{Min}(P_{R,t})}/\text{avg}(\text{Min}(P_{R,t})), 1 + 2.576 s_{\text{Min}(P_{R,t})}/\text{avg}(\text{Min}(P_{R,t}))]$, and $[1 - 2.807 s_{\text{Min}(P_{R,t})}/\text{avg}(\text{Min}(P_{R,t})), 1 + 2.807 s_{\text{Min}(P_{R,t})}/\text{avg}(\text{Min}(P_{R,t}))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.

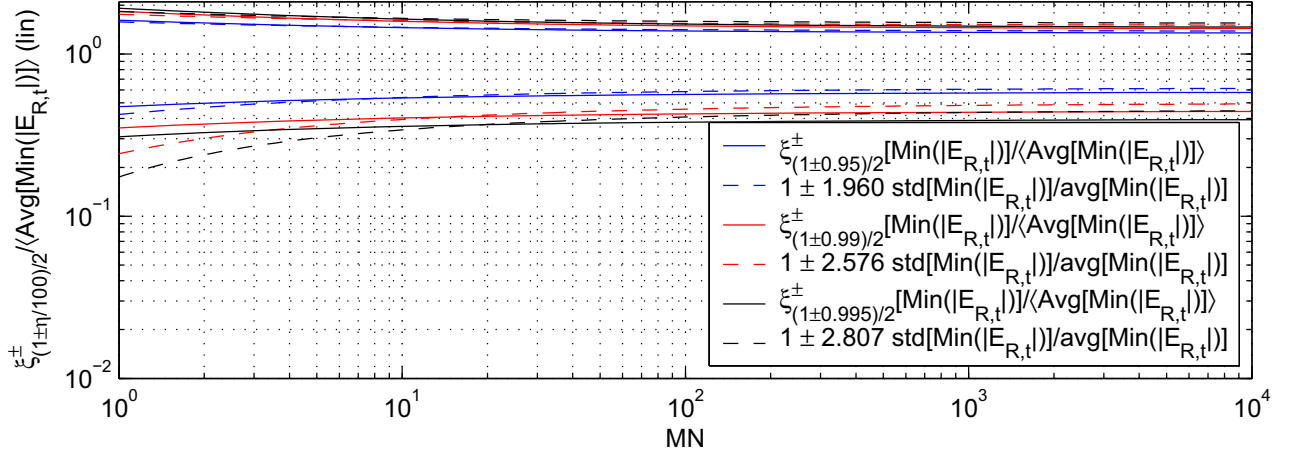


(a)

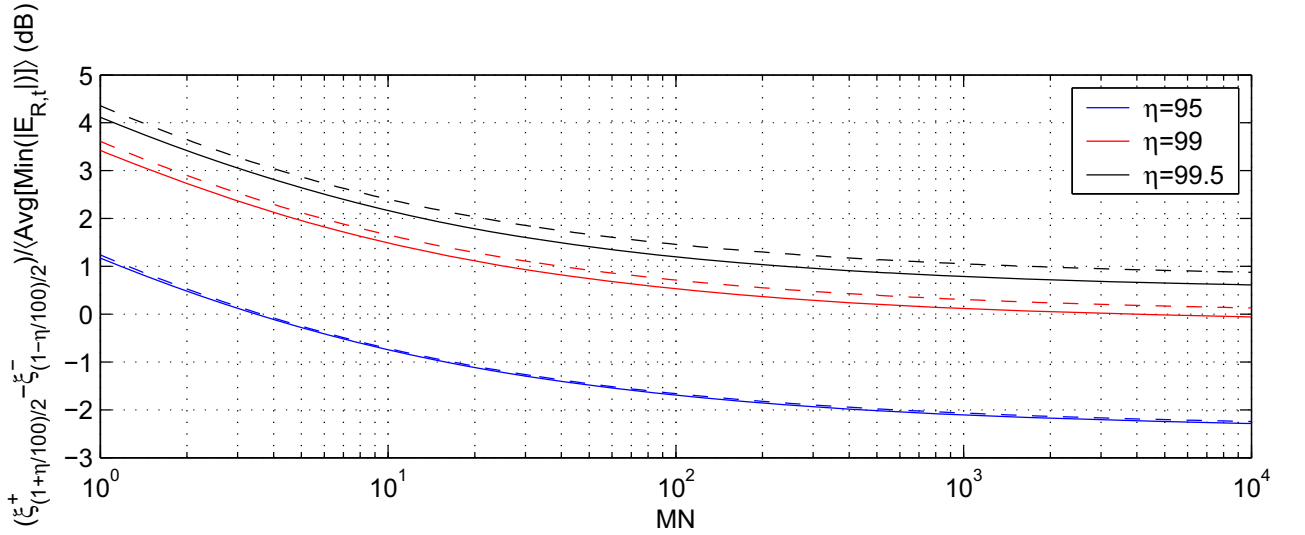


(b)

Figure 45: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Min}(|E_{R,\alpha}|)$ ($p = 1$): (solid lines) exact, based on percentiles of minimum-value distribution of $|E_{R,\alpha}|$ (χ_{2N} parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 \cdot s_{\text{Min}(|E_{R,\alpha}|)} / \text{avg}(\text{Min}(|E_{R,\alpha}|)), 1 + 1.960 \cdot s_{\text{Min}(|E_{R,\alpha}|)} / \text{avg}(\text{Min}(|E_{R,\alpha}|))]$, $[1 - 2.576 \cdot s_{\text{Min}(|E_{R,\alpha}|)} / \text{avg}(\text{Min}(|E_{R,\alpha}|)), 1 + 2.576 \cdot s_{\text{Min}(|E_{R,\alpha}|)} / \text{avg}(\text{Min}(|E_{R,\alpha}|))]$, and $[1 - 2.807 \cdot s_{\text{Min}(|E_{R,\alpha}|)} / \text{avg}(\text{Min}(|E_{R,\alpha}|)), 1 + 2.807 \cdot s_{\text{Min}(|E_{R,\alpha}|)} / \text{avg}(\text{Min}(|E_{R,\alpha}|))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.



(a)



(b)

Figure 46: (a) Mean-normalized boundaries of $\eta\%$ confidence intervals of $\text{Min}(|E_{R,t}|)$ ($p = 3$): (solid lines) exact, based on percentiles of minimum-value distribution of $|E_{R,t}|$ (χ_{6N} parent distribution) with $\eta = 95, 99$, or 99.5 ; (dashed lines) boundaries of Gaussian approximation intervals $[1 - 1.960 \ s_{\text{Min}(|E_{R,t}|)}/\text{avg}(\text{Min}(|E_{R,t}|)), 1 + 1.960 \ s_{\text{Min}(|E_{R,t}|)}/\text{avg}(\text{Min}(|E_{R,t}|))]$, $[1 - 2.576 \ s_{\text{Min}(|E_{R,t}|)}/\text{avg}(\text{Min}(|E_{R,t}|)), 1 + 2.576 \ s_{\text{Min}(|E_{R,t}|)}/\text{avg}(\text{Min}(|E_{R,t}|))]$, and $[1 - 2.807 \ s_{\text{Min}(|E_{R,t}|)}/\text{avg}(\text{Min}(|E_{R,t}|)), 1 + 2.807 \ s_{\text{Min}(|E_{R,t}|)}/\text{avg}(\text{Min}(|E_{R,t}|))]$ for corresponding values of η . (b) Associated widths (mean-normalized differences) of $\eta\%$ confidence intervals, in dB.

7 Conclusion

In this report, we presented theoretical and experimental results on the statistical intrinsic field uncertainty inside a MT/MSRC, originating from the quasi-random nature of the interior field. Sample statistics and their standard errors were calculated for the case of general (including relatively small) numbers of independent stir states N , for single and hybrid stirring and/or scanning. The effect of a finite size of the ensemble (resulting in a bounded maximum number of independent stir states) was investigated. The standard errors show a characteristic decrease with N in accordance with $1/\sqrt{N}$ for large values of N .

With regard to chamber validation and emissions testing, general expressions were derived for the standard error of the average total radiated power in terms of the four S-parameters and the power received from the radiating EUT. In an ideal reverberant environment, the effect of impedance mismatch (S_{11}) on the standard error of the average radiated power can be estimated from transmission (S_{21} -) parameter data only, although expressions for the general case of an imperfect chamber were also given. Using measured emissions data for a real EUT, the width of a 95 % confidence interval for $\text{avg}(P_\alpha)$ is of the order of 2 to 3 dB at 1 GHz.

With regard to immunity testing, for a specified confidence level the confidence interval for the maximum field level is considerably wider and, asymptotically, narrows more slowly with increasing N compared to the confidence interval for the average total emitted power: for the average Cartesian power,

$$s_{\text{Avg}}(P_\alpha) \propto \frac{1}{\sqrt{N}}, \quad s_{\text{Avg}}(|E_\alpha|) \propto \frac{1}{\sqrt{N}} \quad (251)$$

whereas for the maximum Cartesian field strength or power

$$s_{\text{Max}}(P_\alpha) \sim 1, \quad s_{\text{Max}}(|E_\alpha|) \sim \frac{1}{\sqrt{\ln(N)}}. \quad (252)$$

Depending on whether the width of the confidence interval is expressed as a ratio or as a difference of the percentiles, different expressions (199) or (200) are obtained (Fig. 20).

Discrepancies occur with regard to hybrid stirring or (as a special case) when repeating a single stirring process of N independent samples for the EUT at M different locations or orientations. If the maximum value is extracted from the entire $M \times N$ -sized data set, i.e., for a $\chi_{2pMN}^{(2)}$ parent distribution for the power or field strength, then the average maximum field and confidence limits depend on the product of M and N only, i.e., stir and scan states are interchangeable in the estimation of the maximum value. By contrast, if one maximum value for each of the M locations is extracted (e.g., using a max-hold function) and these M sample maxima are used to estimate the overall maximum,

then an asymmetry with respect to the dependence on M and N exists, particularly for relatively small values of M . Thus, the particular design (procedure) of the experiment will, to a certain extent, determine the measurement uncertainty.

The dependence of confidence intervals for the expected value and standard deviation of the maximum power or field strength on both M and N was derived and investigated. With regard to chamber validation, application to the quantification of uncertainty of the IEC 61000-4-21 field nonuniformity limits (Tbl. 2) revealed that the interval for $\sigma_{\text{Max}(|E_{R,\alpha}|)}$ is of the same order of magnitude as the one for $\langle \text{Max}(|E_{R,\alpha}|) \rangle$, underlining that sample variability may have a substantial effect on the outcome of the validation procedure and on the accuracy of field estimation, particularly when N is relatively small. Probability distributions were derived that take the uncertainty of the distribution parameters into account, showing that the latters' uncertainty forms a crucial factor in the evaluation of MU and calling for their individual characterization to minimize MU during validation and testing.

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A Sampling Distributions for Standardized Power and Field Magnitude with Sample-Based Estimated Distribution Parameters

A.1 One-Dimensional Sampling

A.1.1 Power and Field Magnitude

We derive the one-dimensional sampling distribution for the power P_R associated with the complex-valued Cartesian or vectorial electric (or, analogously, magnetic) field, E_R , inside an ideal MT/MSRC, i.e., whose analytic field exhibits a circular Gauss normal distribution.

Since the sample mean and sample standard deviation are independent random variables only in the case of a Gauss normal parent distribution (cf. Sec. 3.1.2), the statistics of their ratio cannot be rigorously derived by classical methods for a χ^2_p ensemble distributed power P_R . Instead, the sampling distribution of $P_R/\text{Std}(P_R)$ is obtained here based on the Student t distribution for $E_R/\text{Std}(E_R)$ and followed by successive variate transformations, as follows.

Since the pdf of E_R is circular, we can investigate its independent and identically distributed (i.i.d.) real and imaginary parts $X \triangleq E'_R \equiv \Re(E_R)$ and $E''_R \equiv \Im(E_R)$ in isolation. In an ideal chamber and in the absence of line-of-sight propagation, the parent (ensemble) distribution of X is a central⁵⁸ Gauss normal distribution ($\langle E_R \rangle = 0$), as we shall further assume. Therefore, the standardized N -point sample variance of X , i.e., $D_{N-1} \triangleq (N-1) S_X^2 / \sigma_X^2 = \sum_{i=1}^N [X_i - \text{Avg}(X)]^2 / \sigma_X^2$ for $N \geq 2$, has a standard χ^2_{N-1} pdf, whereas $D_1 \triangleq [X - \text{Avg}(X)]^2 / \sigma_X^2$ has a standard χ^2_1 pdf. Consequently, the ratio⁵⁹

$$T_X \triangleq \frac{X - \text{Avg}(X)}{S_X} = \sqrt{\frac{D_1}{D_{N-1}/(N-1)}} \quad (253)$$

has a Student t pdf with $N-1$ degrees of freedom representing the sampling pdf of X , i.e.,

$$f_{T_X}(x) \equiv f_X(x; N) = \frac{\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{N-1}{2}\right) \sqrt{N-1}} \left(1 + \frac{x^2}{N-1}\right)^{-N/2} \quad (254)$$

for $N > 1$ (the value of N need not be an integer). The sampling pdf of E'_R or E''_R itself is, approxi-

⁵⁸The Student t sampling distribution also applies in the presence of a line-of-sight component, which yields a deterministic bias $\langle X \rangle = \langle E_R^{(l)} \rangle \neq 0$ in the received field.

⁵⁹The random variable is here X , rather than $\text{Avg}(X)$ as is usual in the classical introduction to the Student t distribution, i.e., here we consider the ratio $[(X - \text{Avg}(X))/\sigma_X]/(S_X/\sigma_X)$ instead of $[(\text{Avg}(X) - \langle X \rangle)/(\sigma_X/\sqrt{N})]/(S_X/\sigma_X)$.

mately⁶⁰, $f_X(x = [e_R^{(\prime)} - \text{avg}(E_R^{(\prime)})]/s_{E_R^{(\prime)}}; N)/s_{E_R^{(\prime)}}$, i.e.,

$$f_{E_R^{(\prime)}}(e_R^{(\prime)}; N) \simeq \frac{\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{N-1}{2}\right) \sqrt{N-1} s_{E_R^{(\prime)}}} \left[1 + \frac{1}{N-1} \left(\frac{e_R^{(\prime)} - \text{avg}(E_R^{(\prime)})}{s_{E_R^{(\prime)}}}\right)^2\right]^{-N/2}. \quad (255)$$

Comparing the sampling pdf (255) to the associated Gauss normal ensemble distribution, $f_{E_R^{(\prime)}}(e_R^{(\prime)}; N \rightarrow +\infty)$, the sampling pdf still has a zero⁶¹ mean value but now with an increased sample standard deviation, viz., $\sqrt{(N-1)/(N-3)} s_{E_R^{(\prime)}} > s_{E_R^{(\prime)}}$ valid for $N > 3$.

The square of the sample value of the standardized X , defined as $Y_1/S_{Y_1} \triangleq T_X^2 = [X - \text{Avg}(X)]^2/S_X^2$, exhibits a χ_1^2 distribution. Its sampling distribution follows from a variate transformation of (254) as a t^2 distribution:

$$f_{Y_1/S_{Y_1}}(y_1/s_{Y_1}; N) = \frac{\Gamma\left(\frac{N}{2}\right)}{2\sqrt{\pi} \Gamma\left(\frac{N-1}{2}\right) \sqrt{N-1}} \left[\frac{y_1}{s_{Y_1}} \left(1 + \frac{y_1}{(N-1)s_{Y_1}}\right)^N\right]^{-1/2}. \quad (256)$$

This represents a $F_{1,N-1}$ distribution (i.e., Fisher–Snedecor F distribution with $(1, N-1)$ degrees of freedom), because T_X^2 is a ratio of respective standard χ_1^2 -distributed and χ_{N-1}^2 -distributed variates D_1 and D_{N-1} that are both statistically independent (because $\text{Avg}(X)$ and S_X are independent for Gauss normal real X [11]) and whereby each variate is divided by its number of degrees of freedom:

$$t^2 = F_{1,N-1} = \frac{\chi_1^2/1}{\chi_{N-1}^2/(N-1)}. \quad (257)$$

To find the sampling distribution of a squared Cartesian or vectorial EM field with $2p$ i.i.d. components Y_i (corresponding to a p -dimensional analytic circular complex-valued field), i.e., $\sigma_{Y_i} = \sigma_Y = \sigma_X^2$, consider the ratio $U \triangleq Z/K_Z$, where

$$Z \triangleq \frac{Y}{\sigma_Y} = \sum_{i=1}^{2p} \frac{Y_i}{\sigma_Y} = \sum_{i=1}^{2p} \frac{[X_i - \text{avg}(X)]^2}{\sigma_X^2} \quad (258)$$

has a standard χ_{2p}^2 distribution, on account of the addition theorem for $2p$ variates, each with i.i.d. standard χ_1^2 distributions, and

$$K_Z \triangleq \frac{(2pN-1)\text{Var}(X)}{\sigma_X^2} \quad (259)$$

exhibits a standard χ_{2pN-1}^2 distribution, with $\text{Var}(X) \equiv S_X^2 = \sum_{i=1}^{2pN} [X_i - \text{Avg}(X)]^2/(2pN-1)$. Hence, U is a scaled Fisher–Snedecor $F_{2p,2pN-1}$ variate:

$$U \triangleq \frac{Z}{K_Z} \equiv \frac{2p}{2pN-1} \frac{Z/(2p)}{K_Z/(2pN-1)} = \frac{2p}{2pN-1} F_{2p,2pN-1}. \quad (260)$$

⁶⁰In (255) and corresponding expressions (268) and (278), the approximation is due to the fact that $s_{E_R^{(\prime)}}$ is here treated as a (constant) parameter, which implies requiring a sufficiently small value of $\sigma_{\text{Std}(E_R^{(\prime)})}$. An exact result will be derived in [44]. However, the discrepancy is negligible for most practical purposes ($N > 5$).

⁶¹for the case when a line-of-sight component is present: cf. preceding footnote

With the aid of (10), the ratio Z/K_Z can be related to the standard sampled received power (or intensity) as

$$\frac{P_R}{S_{P_R}} = \frac{\sum_{i=1}^{2p} (E'_{R,i})^2}{2\sqrt{p} S_{E'_R}^2} = \frac{2pN-1}{2\sqrt{p}} \frac{\sum_{i=1}^{2p} (E'_{R,i}/\sigma_{E'_R})^2}{(2pN-1) \text{Var}(E'_R)/\sigma_{E'_R}^2} = \frac{2pN-1}{2\sqrt{p}} \frac{Z}{K_Z}. \quad (261)$$

Thus, combining (260) and (261),

$$W \triangleq \frac{P_R}{S_{P_R}} \equiv \frac{|E_R|^2}{S_{|E_R|^2}} = \sqrt{p} F_{2p, 2pN-1}. \quad (262)$$

With the definition of U in (260), upon scaling the standard $F_{2p, 2pN-1}$ distribution,

$$f_U(u; N) = \frac{\Gamma\left[p(N+1) - \frac{1}{2}\right]}{\Gamma(p)\Gamma(pN - \frac{1}{2})} \frac{(2p)^p (2pN-1)^{pN-\frac{1}{2}}}{\frac{2p}{2pN-1}} \frac{\left(\frac{2pN-1}{2p} u\right)^{p-1}}{\left[2pN-1 + 2p\left(\frac{2pN-1}{2p} u\right)\right]^{p(N+1)-\frac{1}{2}}} \quad (263)$$

$$= \frac{\Gamma\left(pN - \frac{1}{2} + p\right)}{\Gamma(pN - \frac{1}{2})\Gamma(p)} \frac{u^{p-1}}{(1+u)^{pN-\frac{1}{2}+p}}. \quad (264)$$

For the square of the Cartesian in-phase field component $[\Re(E_R)]^2$ or the quadrature component $[\Im(E_R)]^2$ (i.e., for $p = 1/2$), (264) reduces to (256), as expected. Finally, from (262), the sought sampling distribution of $W = P_R/S_{P_R}$ at $w = p_R/s_{P_R}$ follows as

$$f_W(w; N) = \frac{\Gamma\left[p(N+1) - \frac{1}{2}\right]}{\Gamma(p)\Gamma(pN - \frac{1}{2})} \frac{(2p)^p (2pN-1)^{pN-\frac{1}{2}}}{\sqrt{p}} \frac{\left(\frac{w}{\sqrt{p}}\right)^{p-1}}{\left[2pN-1 + 2p\left(\frac{w}{\sqrt{p}}\right)\right]^{p(N+1)-\frac{1}{2}}} \quad (265)$$

$$= \frac{\Gamma\left(pN - \frac{1}{2} + p\right)}{\Gamma(pN - \frac{1}{2})\Gamma(p)} \frac{p^{p/2}}{\left(pN - \frac{1}{2}\right)^p} \frac{w^{p-1}}{\left(1 + \frac{\sqrt{p}}{pN-\frac{1}{2}} w\right)^{pN-\frac{1}{2}+p}}. \quad (266)$$

For a sample-based estimated s_{P_R} , the pdf for the sampled power itself is then, approximately⁶²,

$$f_{P_R}(p_R; N) \simeq f_W(w=p_R/s_{P_R}; N) / s_{P_R} \quad (267)$$

$$\simeq \frac{\Gamma\left(pN - \frac{1}{2} + p\right)}{\Gamma(pN - \frac{1}{2})\Gamma(p)} \frac{p^{p/2}}{\left(pN - \frac{1}{2}\right)^p s_{P_R}} \frac{\left(\frac{p_R}{s_{P_R}}\right)^{p-1}}{\left(1 + \frac{\sqrt{p}}{pN-\frac{1}{2}} \frac{p_R}{s_{P_R}}\right)^{pN-\frac{1}{2}+p}}. \quad (268)$$

By way of verification, we consider the limit of this sampling pdf when $N \rightarrow +\infty$. In this limit, the exponent $p(N+1) - \frac{1}{2}$ reduces to $pN - \frac{1}{2}$; the prefactor $\Gamma(pN - \frac{1}{2} + p)/\Gamma(pN - \frac{1}{2}) = (pN - \frac{1}{2}) \cdot \dots \cdot (pN - \frac{3}{2} + p)$ for $p \geq 1$ becomes $(pN - \frac{1}{2})^p$; and s_{P_R} approaches σ_{P_R} . Hence, taking the limit $N \rightarrow +\infty$ results in

$$f_{P_R}(p_R) \rightarrow \frac{p^{p/2}}{\Gamma(p)\sigma_{P_R}} \left(\frac{p_R}{\sigma_{P_R}}\right)^{p-1} \exp\left(-\frac{\sqrt{p} p_R}{\sigma_{P_R}}\right) \quad (269)$$

⁶²see previous footnote

which is the ensemble χ_{2p}^2 limit pdf (8), as expected. It also satisfies a well-known limit theorem from probability theory stating that the cdf $F_{m,n}(x)$ converges⁶³ to the cdf $\chi_m^2(mx)$ when $n/m \rightarrow +\infty$, i.e., $F_q(m, n) \rightarrow \chi_q^2(m)/m$ for the corresponding quantiles. In the limit $N \rightarrow +\infty$, the pdf of the F variate $F_{2p, 2pN-1}$ exhibits the asymptotic mean value

$$\langle F_{2p, 2pN-1} \rangle = \frac{2pN-1}{2pN-3} \rightarrow 1 \quad (270)$$

and sample variance (valid for $2pN > 5$)

$$\sigma_{F_{2p, 2pN-1}}^2 = \frac{2(2pN-1)^2(2pN+2p-3)}{2p(2pN-3)^2(2pN-5)} \rightarrow \frac{1}{p}. \quad (271)$$

Correspondingly, $\langle W \rangle \equiv \sqrt{p} \langle F_{2p, 2pN-1} \rangle \rightarrow \sqrt{p}$ and $\sigma_W \equiv \sqrt{p} \sigma_{F_{2p, 2pN-1}} \rightarrow 1$, whereas $\langle P_R \rangle \rightarrow \sqrt{p} \sigma_{P_R}$.

Thus, we retrieve the coefficient of variation of P_R and its asymptotic value as

$$\frac{\sigma_{P_R}}{\langle P_R \rangle} = \sqrt{\frac{2pN+2p-3}{p(2pN-5)}} \rightarrow \frac{1}{\sqrt{p}} \quad (272)$$

i.e., the value for $N \rightarrow +\infty$ is indeed that for the asymptotic χ_{2p}^2 ensemble distribution, whereas for small N its value is substantially larger, indicating larger relative uncertainty.

For the standardized field magnitude

$$V \triangleq \frac{|E_R|}{S_{|E_R|}} = \sqrt{\frac{S_V^2}{S_W}} W \quad (273)$$

and recalling that

$$\frac{s_V^2}{s_W} = \frac{\sigma_{|E_R|}^2}{\sigma_{|E_R|^2}} = \frac{p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2}{\sqrt{p}} \quad (274)$$

for $\chi_{2p}^{(2)}$ distributions, from variate transformation of (266) and noting that

$$\frac{w}{s_W} = \left(\frac{v}{s_V} \right)^2 = \frac{v^2}{s_W} \cdot \frac{s_W}{s_V^2}, \quad dw = 2v(s_W/s_V^2)dv, \quad (275)$$

we obtain the sampling pdf of V at $v = |e_R|/s_{|E_R|}$ when using a sample-based estimate $s_{|E_R|}$ for the sample standard deviation $S_{|E_R|}$ as

$$f_V(v; N) = \frac{\Gamma(pN - \frac{1}{2} + p)}{\Gamma(pN - \frac{1}{2})\Gamma(p)} \frac{4}{(pN - \frac{1}{2})^p} \left[p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2 \right]^p \frac{v^{2p-1}}{\left[1 + \frac{p - \left(\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} \right)^2}{pN - \frac{1}{2}} v^2 \right]^{pN - \frac{1}{2} + p}} \quad (276)$$

⁶³For example, from Table 7 in Appendix C and Table 8 in Appendix D for the percentiles of the respective distributions, $F_{0.025}(m=6, n \rightarrow +\infty) = 0.2062240 \dots = 1.2373442 \dots / 6 = \chi_{0.025}^2(M-1=6)/6$.

which we can refer to as a root- F distribution. Thus, the pdf for the standardized sampled field magnitude itself is then, approximately⁶⁴,

$$f_{|E_R|}(|e_R|; N) \simeq f_V(v=|e_R|/s_{|E_R|}; N) / s_{|E_R|} \quad (277)$$

$$\simeq \frac{\Gamma(pN - \frac{1}{2} + p)}{\Gamma(pN - \frac{1}{2})\Gamma(p)} \frac{4}{(pN - \frac{1}{2})^p s_{|E_R|}} \left[p - \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \right)^2 \right]^p \frac{\left(\frac{|e_R|}{s_{|E_R|}} \right)^{2p-1}}{\left[1 + \frac{p - \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \right)^2}{pN - \frac{1}{2}} \left(\frac{|e_R|}{s_{|E_R|}} \right)^2 \right]^{pN - \frac{1}{2} + p}}. \quad (278)$$

In the limit $N \rightarrow +\infty$, with limit approximations similar to those given above, (278) reduces to

$$f_{|E_R|}(|e_R|) = \frac{2 \left[p - \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \right)^2 \right]^p}{\Gamma(p) \sigma_{|E_R|}} \left(\frac{|e_R|}{\sigma_{|E_R|}} \right)^{2p-1} \exp \left\{ - \left[p - \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)} \right)^2 \right] \left(\frac{|e_R|}{\sigma_{|E_R|}} \right)^2 \right\} \quad (279)$$

i.e., the χ_{2p} pdf (19), as expected.

A.1.2 Average Value of Power and Field Magnitude

The sampling distributions (255), (268) and (278) serve as a basis for deriving distributions of specific sample statistics of interest for the complex field (or current), power and amplitude. For example, if instead of P_R/S_{P_R} we are interested in the N -point sample average $\text{Avg}(P_R)/S_{\text{Avg}(P_R)}$, the previous results can be formally repeated. Since $\text{Avg}(E_R)$ is again circular Gauss normal, its sampling distribution is again a Student t given by (255), but with $s_{E_R^{(\prime)}}$ re-scaled to $s_{\text{Avg}(E_R^{(\prime)})} = s_{E_R^{(\prime)}}/\sqrt{N}$. Although F distributions, unlike χ^2 distributions, do not possess an additivity property, in general, it appears that averaging based on incoherent superposition of the random field produces again an F distribution. In essence, this is because the nature of the χ_{2pN-1}^2 distribution of the denominator of the F ratio remains unchanged for averaging. Specifically, from (3), (6), and (10),

$$s_{\text{Avg}(P_R)} = \frac{\sigma(\sum_{i=1}^N P_{R,i})}{N} = \frac{\sigma_{P_R}}{\sqrt{N}} = 2\sqrt{\frac{p}{N}} \sigma_{E_{R,\alpha}'}^2 \simeq \frac{2\sqrt{pN}}{N-1} s_{E_{R,\alpha}'}^2 \quad (280)$$

and instead of (261) we obtain the standardized sample-averaged power as

$$R \triangleq \frac{\text{Avg}(P_R)}{S_{\text{Avg}(P_R)}} = \frac{(1/N) \sum_{i=1}^{pN} [(E'_{R,\alpha,i})^2 + (E''_{R,\alpha,i})^2]}{2\sqrt{pN} S_{E_{R,\alpha}'}^2 / (N-1)} \quad (281)$$

$$= \frac{2pN-1}{2\sqrt{pN}} \frac{N-1}{N} \frac{\sum_{i=1}^{pN} [(E'_{R,\alpha,i}/\sigma_{E_{R,\alpha}'}')^2 + (E''_{R,\alpha,i}/\sigma_{E_{R,\alpha}'}'')^2]}{(2pN-1) \text{Var}(E'_{R,\alpha})/\sigma_{E_{R,\alpha}'}^2}. \quad (282)$$

Compared to (260), the number of degrees of freedom of the numerator in (282) has increased by a factor N as a result of aggregation in the calculation of the arithmetic mean. Hence

$$R = \sqrt{\frac{p}{N}} (N-1) F_{2pN, 2pN-1} \quad (283)$$

⁶⁴see previous footnote

compared to (262). The number of degrees of freedom of the numerator is now larger than (but asymptotically equal to) that of the denominator. Thus, for arbitrary N , the pdf of R is

$$f_R(r; N) = \frac{\Gamma(2pN - \frac{1}{2})}{\Gamma(pN)\Gamma(pN - \frac{1}{2})} \frac{(2pN)^{pN} (2pN - 1)^{pN - \frac{1}{2}}}{\sqrt{\frac{p}{N}} (N - 1)} \frac{\left(\sqrt{\frac{N}{p}} \frac{r}{N-1}\right)^{pN-1}}{\left(2pN - 1 + 2pN\sqrt{\frac{N}{p}} \frac{r}{N-1}\right)^{2pN - \frac{1}{2}}} \quad (284)$$

$$= \frac{\Gamma(2pN - \frac{1}{2})}{\Gamma(pN)\Gamma(pN - \frac{1}{2})} \left[\frac{2\sqrt{pN^3}}{(2pN - 1)(N - 1)} \right]^{pN} \frac{r^{pN-1}}{\left(1 + \frac{2\sqrt{pN^3}}{(2pN-1)(N-1)} r\right)^{2pN - \frac{1}{2}}}. \quad (285)$$

A.1.3 Maximum and Minimum Value of Power and Field Magnitude

The sampling distributions of P_R/S_{P_R} and $|E_R|/S_{|E_R|}$ may also be used to derive asymptotic sample distributions for the sample maximum or minimum value, for an estimate of the standardized sample standard deviation derived from the sample itself. To this end, we require expressions for their sampling *cdfs* F in F^N or $1 - (1 - F)^N$, respectively. After some calculation, with the aid of [25, (8.380.3) & (8.391)], these cdfs follow as

$$F_W(w; N) = I_{\phi(w)}\left(p, pN - \frac{1}{2}\right), \quad F_V(v; N) = I_{\psi(v)}\left(p, pN - \frac{1}{2}\right) \quad (286)$$

with

$$\phi(w) \triangleq \frac{1}{1 + \frac{pN - \frac{1}{2}}{\sqrt{p} w}}, \quad \psi(v) \triangleq \frac{1}{1 + \frac{pN - \frac{1}{2}}{\left[p - \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p)}\right)^2\right] v^2}} \quad (287)$$

where

$$I_x(u, v) \equiv \frac{B_x(u, v)}{B(u, v)} \triangleq \frac{1}{B(u, v)} \int_0^x t^{u-1} (1-t)^{v-1} dt = \frac{1}{B(u, v)} \int_0^{\frac{1}{x-1}} \frac{t^{u-1}}{(1+t)^{u+v}} dt \quad (288)$$

and

$$B(u, v) \triangleq \int_0^1 t^{u-1} (1-t)^{v-1} dt = \int_0^{+\infty} \frac{t^{u-1}}{(1+t)^{u+v}} dt \quad (289)$$

represent regularized incomplete and complete beta functions, respectively (cf. [25, eqs. (8.380.1–3) & (8.391–8.392)]). Note that the upper and lower limits of a two-sided $\eta\%$ confidence interval for an F distribution are related via

$$F_{m,n}\left(\frac{1 + \eta/100}{2}\right) = \frac{1}{F_{n,m}\left(\frac{1 - \eta/100}{2}\right)}. \quad (290)$$

For reference, Table 8 in Appendix D lists calculated percentile values for one- and two-sided 95% confidence intervals for Cartesian and vectorial $F_{2p, 2pN-1}$ fields⁶⁵.

⁶⁵Apart from the cases $m \equiv 2p = 2$ and $m = 6$, the Table also lists values for $m = 4$, which is relevant to sampling distributions for boundary random fields, whether propagating in a single direction (e.g., unpolarized light in the paraxial approximation) or for isotropic 3-D EM fields near a planar surface [23], [24].

A.2 Hierarchical Two-Dimensional (Space-Time) Sampling

A.2.1 General Remarks

Exact or asymptotic sampling distributions form a platform for repeated sampling, producing $M-1$ additional sets of N points. If the original first set of N samples produced by a single stirrer (mechanical or electronic) is conceived as a sampling in time (i.e., at different stir states), then the secondary, tertiary, etc., samplings may be samplings in space (i.e., at different spatial locations), although multiple time dimensions for the sampling space are also possible (different paddle wheel stirrers; successive mechanical and frequency stirring, etc.). Likewise, the sampling space can be made up of different spatial directions or polarizations, etc. We shall focus attention and discussion to two-dimensional time-space or time-time sampling.

When the numbers of degrees of freedom in each dimension, N and M , are large compared to 1, then an argument on the basis of ergodicity may be used to assert that the secondary and any higher-order sampling are statistically equivalent (i.e., produce the same sample statistics) as the original one-dimensional stir process. When M or N are not exceedingly large, this equivalence will not hold in general. In the latter case, the multi-dimensional sampling must be analyzed explicitly. A hierarchical sampling scheme can be conceived, whereby the first sample and its corresponding sample distribution (in time) serves as the “parent” distribution and pool for sampling in the second (spatial) dimension. Again, an issue arises with regard to dependence between the sample average and sample standard deviation for non-Gaussian parent distributions (cf. Sec. 3.1.3). Therefore, (i) the nonlinear square-law transformation from the Gauss normal analytic field to its magnitude or power must be delayed until the end, and (ii) only the case $N \rightarrow +\infty$ for finite M can be rigorously solved, but reduces to the previous analysis.

In order to make progress while keeping the calculations tractable, we derive an *approximate* distribution based on a hierarchical 2-D space-time sampling joint distribution. One justification for this approach is that we are interested in the effect of the *ratio* of the parameters when both of them approach infinity at *different* rates, in which case the approximate result can be expected to become increasingly accurate again.

A.2.2 Analysis

In the 2-D sampling plane, we consider sampling across M points in space and N instances in time, producing a total of $M \times N$ sample values (cf. Fig. 21). In a hierarchical sampling scheme, we consider N (i.e., the number of sample points within one set) to produce the lower level sampling and M (i.e.,

the number of such sample sets) to yield the higher level sampling. For $M \rightarrow +\infty$ or $N \rightarrow +\infty$, the sampling reduces to 1-D sampling in the other dimension.

While it would be advisable⁶⁶ to start with the analysis for the complex field, we shall immediately consider the squared field magnitude or power itself as our variates of interest, in order to keep the mathematical analysis tractable. To this end, consider the χ_{2pN}^2 distribution as an approximate sampling distribution for the 1-D time sampling:

$$f_X(x; N) = \frac{x^{pN-1} \exp\left(-\frac{\sqrt{pN} x}{\sigma_X}\right)}{\Gamma(pN) (\sigma_X/\sqrt{pN})^{pN}} \quad (291)$$

with $\langle X \rangle = 2pN\sigma_{E_R}^2$ and $\sigma_X = 2\sqrt{pN}\sigma_{E_R}^2$; cf. (11)–(12). Each N -dimensional sample set generates one sample value for use in the sampling in the M -dimension. Upon collecting the resulting M samples, the expected value of the sample average remains unchanged, i.e. $\text{Avg}(X)$: for sufficiently large M and N ,

$$\begin{aligned} \frac{1}{MN} \sum_{k=1}^M \sum_{\ell=1}^N P_R(k, \ell) &\equiv \frac{1}{M} \sum_{k=1}^M \left(\frac{1}{N} \sum_{\ell=1}^N P_R(k, \ell) \right) \simeq \frac{1}{M} \sum_{k=1}^M \text{avg}_N[P_R(k, \cdot)] \\ &\simeq \frac{1}{N} \sum_{\ell=1}^N P_R(k_0, \ell), \quad \forall k_0 \end{aligned} \quad (292)$$

whence

$$\langle X \rangle = \frac{2p MN \sigma_{E_R}^2}{MN} = 2p \sigma_{E_R}^2. \quad (293)$$

By contrast, the standard error of the average will scale by a further factor $1/\sqrt{M}$, hence

$$\sigma_{\text{Avg}(X)} = 2\sqrt{\frac{p}{NM}} \sigma_{E_R}^2. \quad (294)$$

For $M \rightarrow +\infty$, the sample variance exhibits a χ_{MN}^2 distribution.

Defining the F ratio

$$H \triangleq \frac{X/(2pM)}{\Sigma_X/(2MN)} \quad (295)$$

and assuming the sample mean and sample standard deviation to be sufficiently weakly independent (cf. *supra*), then H is an F ratio and the joint pdf of the sample mean and sample standard deviation is approximately equal to the product of their marginal pdfs, i.e.,

$$f_{X, \Sigma_X}(x, \sigma_X) \simeq \left[\frac{(pM)^{pM/2}}{\Gamma(pM)} x^{pM-1} \exp\left(-\sqrt{pM}x\right) \right] \left[\frac{(MN)^{MN/2}}{\Gamma(MN)} \sigma_X^{MN-1} \exp\left(-\sqrt{MN}\sigma_X\right) \right] \quad (296)$$

⁶⁶The Student t distribution approaches Gauss normality more rapidly than a χ_{2N}^2 , which would make the approximation of statistical independence of the 1-D sample mean and sample variance even better justified.

i.e., the product of a χ^2_{2pM} pdf for $\sigma_X = 1$ and a χ^2_{2MN} pdf with $\sigma_{\Sigma_X} = 1$. After some calculation and with the aid of [25, eqs. (3.381.4), (3.194.3) & (8.384.1)],

$$f_H(h) = \frac{\Gamma[M(p+N)]}{\Gamma(pM)\Gamma(MN)} \frac{(2pM)^{pM}}{(2MN)^{MN}} \frac{h^{pM-1}}{(2pM + 2MN h)^{M(p+N)}}. \quad (297)$$

It is emphasized that this result is asymptotic, in the sense that its strict validity requires both $N \rightarrow +\infty$ (in order that the time-domain sampling distribution of X can be assumed to be sufficiently close to χ^2_{2pN} pdf, although its approach to Gauss normality is not required when considering the F ratio⁶⁷) and $M \rightarrow +\infty$ (in order that Σ_X^2 approaches a χ^2 distribution with an infinite number of degrees of freedom, i.e., a Gauss normal distribution as well). Only if both conditions are met are the sample average X and sample standard deviation $\Sigma_X/\sqrt{M}$ statistically independent, in which case writing the expression (296) is justified.

Other sampling strategies (groupings, hierarchies) can be devised in the sampling plane, each with there corresponding pairs of degrees of freedom as appropriate, which produce different types of asymptotic F distributions.

⁶⁷Such would be necessary if a t ratio were used for transforming the joint pdf.

B Appendix: Percentiles of Student t Distribution

Table 6: 95% and 99% percentiles $t_q(M-1)$ for Student t distribution with $M-1$ degrees of freedom. For $M \rightarrow +\infty$, we have $t_q(M-1) \rightarrow z_q \sqrt{(M-1)/[(M-1)-2]}$.

$M-1$	$t_{0.995} = -t_{0.005}$	$t_{0.975} = -t_{0.025}$
1	63.66	12.71
2	9.92	4.30
3	5.84	3.18
4	4.60	2.78
5	4.03	2.57
6	3.71	2.45
7	3.50	2.36
8	3.36	2.31
9	3.25	2.26
10	3.17	2.23
11	3.11	2.20
12	3.06	2.18
13	3.01	2.16
14	2.98	2.14
15	2.95	2.13
16	2.92	2.12
17	2.90	2.11
18	2.88	2.10
19	2.86	2.093
20	2.84	2.086
21	2.83	2.080
22	2.82	2.074
23	2.81	2.069
24	2.80	2.064
25	2.79	2.060
26	2.779	2.056
27	2.771	2.052
28	2.763	2.048
29	2.756	2.045
30	2.750	2.042
32	2.738	2.037
34	2.728	2.034
36	2.719	2.028
38	2.712	2.024
40	2.704	2.021
42	2.698	2.018
44	2.692	2.015
46	2.687	2.012
48	2.682	2.010
50	2.678	2.009
60	2.660	2.000
70	2.648	1.994
80	2.639	1.990
90	2.632	1.987
100	2.626	1.984
200	2.601	1.972
300	2.592	1.968
400	2.588	1.966
500	2.586	1.965
600	2.584	1.964
700	2.583	1.9634
800	2.5820	1.9629
900	2.5813	1.9626
1 000	2.5808	1.9623
2 000	2.5783	1.9612
4 000	2.5775	1.9606
6 000	2.5771	1.9604
8 000	2.5764	1.9603
10 000	2.57632	1.96020
10^5	2.57587(8)	1.95998(8)
10^6	2.57583(4)	1.95996(6)
10^8	2.57582(9)	1.95996(4)
$+\infty$	2.57582(9)	1.95996(4)

C Appendix: Percentiles of Helmert χ^2 Distribution

Table 7: 95% and 99% percentiles $\chi_q^2(M-1)$ of Helmert χ^2 distribution with $M-1$ degrees of freedom. For $M \rightarrow +\infty$, we have $\chi_q^2(M-1) \rightarrow (M-1)[1 - \frac{2}{9(M-1)} + z_q \sqrt{\frac{2}{9(M-1)}}]^3 \simeq [z_q + \sqrt{2(M-1)-1}]^2/2$.

$M-1$	$\chi_{0.995}^2$	$\chi_{0.975}^2$	$\chi_{0.025}^2$	$\chi_{0.005}^2$
1	7.88	5.02	.000982	.0000393
2	10.596634734	7.377758908	.050635616	.010025084
3	12.8	9.35	.216	.0717
4	14.860259006	11.143286782	.484418557	.206989094
5	16.7	12.8	.831	.412
6	18.547584180	14.449375336	1.237344246	.675726778
7	20.3	16.0	1.69	.989
8	22.0	17.5	2.18	1.34
9	23.6	19.0	2.70	1.73
10	25.2	20.5	3.25	2.16
11	26.8	21.9	3.82	2.60
12	28.3	23.3	4.40	3.07
13	29.8	24.7	5.01	3.57
14	31.3	26.1	5.63	4.07
15	32.8	27.5	6.26	4.60
16	34.3	28.8	6.91	5.14
17	35.7	30.2	7.56	5.70
18	37.2	31.5	8.23	6.26
19	38.6	32.9	8.91	6.84
20	40.0	34.2	9.59	7.43
21	41.4	35.5	10.3	8.03
22	42.8	36.8	11.0	8.64
23	44.2	38.1	11.7	9.26
24	45.6	39.4	12.4	9.89
25	46.9	40.6	13.1	10.5
26	48.3	41.9	13.8	11.2
27	49.6	43.2	14.6	11.8
28	51.0	44.5	15.3	12.5
29	52.3	45.7	16.0	13.1
30	53.7	47.0	16.8	13.8
35	60.3	53.2	20.6	17.2
40	66.8	59.3	24.4	20.7
45	73.2	65.4	28.4	24.3
50	79.5	71.4	32.4	28.0
55	85.7	77.4	36.4	31.7
60	92.0	83.3	40.5	35.5
65	98.1	89.2	44.6	39.4
70	104.2	95.0	48.8	43.3
75	110.3	100.8	52.9	47.2
80	116.3	106.6	57.2	51.2
85	122.3	112.4	61.4	55.2
90	128.3	118.1	65.6	59.2
95	134.2	123.9	65.9	63.2
100	140.2	129.6	74.2	67.3
150	198.4	185.8	118.0	109.1
200	255.3	241.1	162.7	152.2
250	311.3	295.7	208.1	196.2
300	366.8	349.9	253.9	240.7
350	421.9	403.7	300.1	285.6
400	476.6	457.3	346.5	330.9
450	531.0	510.7	393.1	376.5
500	585.2	563.9	439.9	422.3
550	639.2	616.9	486.9	468.3
600	693.0	669.8	534.0	514.5
650	746.6	722.5	581.2	560.9
700	800.1	775.2	628.6	607.4
750	853.5	827.8	676.0	654.0
800	906.8	880.3	723.5	700.7
850	960.0	932.7	771.1	747.5
900	1013.0	985.0	818.8	794.5
950	1066.0	1037.3	866.5	841.5
1 000	1118.9	1089.5	914.3	888.6
2 000	2166.7	2125.8	1877.9	1840.8
3 000	3203.3	3153.7	2850.1	2804.2
4 000	4234.1	4177.2	3826.6	3773.4
5 000	5261.3	5198.9	4805.9	4746.2
6 000	6285.9	6216.6	5787.2	5721.6
7 000	7308.5	7233.8	6770.0	6699.0
8 000	8329.6	8249.8	7754.0	7677.9
9 000	9349.3	9264.8	8738.9	8658.2
10^4	10 368	10 279	9724.7	9639.5
10^5	101 156	100 878	99 125	98 852
10^6	1 003 647	1 002 774	997 230	996 361
10^7	10 011 529	10 877 015	9 991 240	9 988 490
10^8	100 036 458	100 027 735	99 972 299	99 963 602
10^9	1 000 115 291	1 000 087 779	999 912 403	999 884 901

D Appendix: Percentiles of Fisher-Snedecor F Distribution

Table 8: 95 % (for one-sided interval), 2.5 %, and 97.5 % (for two-sided interval) percentiles $F_q(m, n)$ for F distribution with (m, n) degrees of freedom, defined as $F_{m,n} = (\chi_m^2/m) / (\chi_n^2/n)$. For $n/m \rightarrow +\infty$, we have $F_{m,n}(x) \rightarrow \chi_m^2(mx)$, i.e., $F_q(m, n) \rightarrow \chi_q^2(m)/m$.

n	$\eta = 0.95$ (one-sided)			$(1 + \eta/100)/2 = 0.975$ (two-sided)			$(1 - \eta/100)/2 = 0.025$ (two-sided)		
	$m = 2$	$m = 4$	$m = 6$	$m = 2$	$m = 4$	$m = 6$	$m = 2$	$m = 4$	$m = 6$
1	199.500	224.583	233.986	799.500	899.583	937.112	0.025970	0.08185	0.1135
2	19.000	19.247	19.330	39.000	39.248	39.332	0.025641	0.09390	0.1377
3	9.552	9.117	8.941	16.044	15.101	14.735	0.025533	0.10021	0.1515
4	6.944	6.388	6.163	10.649	9.605	9.198	0.025479	0.10412	0.1606
5	5.786	5.192	4.950	8.434	7.388	6.978	0.025446	0.10679	0.1670
6	5.143	4.534	4.284	7.260	6.227	5.820	0.025425	0.10873	0.1718
7	4.737	4.120	3.866	6.541	5.523	5.119	0.025410	0.11020	0.1756
8	4.459	3.838	3.581	6.060	5.053	4.652	0.025398	0.11136	0.1786
9	4.257	3.633	3.374	5.715	4.718	4.320	0.025389	0.11230	0.1810
10	4.103	3.478	3.217	5.457	4.468	4.072	0.025382	0.11307	0.1831
11	3.982	3.357	3.095	5.256	4.275	3.881	0.025376	0.11372	0.1848
12	3.885	3.259	2.996	5.096	4.121	3.728	0.025371	0.11427	0.1864
13	3.806	3.179	2.915	4.965	3.996	3.604	0.025368	0.11475	0.1876
14	3.789	3.112	2.848	4.857	3.892	3.501	0.025364	0.11516	0.1888
15	3.682	3.056	2.791	4.765	3.804	3.415	0.025361	0.11552	0.1898
16	3.634	3.007	2.741	4.687	3.729	3.341	0.025358	0.11584	0.1907
17	3.592	2.965	2.699	4.619	3.665	3.277	0.025356	0.11613	0.1915
18	3.555	2.928	2.661	4.560	3.608	3.221	0.025353	0.11638	0.1922
19	3.522	2.895	2.628	4.508	3.559	3.172	0.025352	0.11661	0.1929
20	3.493	2.866	2.599	4.461	3.515	3.128	0.025350	0.11682	0.1935
22	3.443	2.817	2.549	4.383	3.440	3.055	0.025347	0.11719	0.1954
24	3.403	2.776	2.508	4.319	3.379	2.995	0.025345	0.11750	0.1954
26	3.369	2.743	2.474	4.266	3.329	2.945	0.025342	0.11776	0.1962
28	3.340	2.714	2.445	4.221	3.286	2.903	0.025341	0.11799	0.1968
30	3.316	2.690	2.420	4.183	3.250	2.867	0.025339	0.11819	0.1974
32	3.295	2.668	2.399	4.149	3.218	2.836	0.025338	0.11836	0.1979
34	3.276	2.645	2.380	4.120	3.191	2.808	0.025337	0.11851	0.1984
36	3.259	2.634	2.364	4.094	3.167	2.785	0.025336	0.11865	0.1988
38	3.245	2.619	2.349	4.071	3.145	2.763	0.025335	0.11879	0.1992
40	3.232	2.606	2.336	4.051	3.126	2.744	0.025334	0.11889	0.1995
42	3.220	2.594	2.324	4.033	3.109	2.727	0.025333	0.11899	0.1998
44	3.209	2.584	2.313	4.016	3.093	2.712	0.025332	0.11909	0.2001
46	3.200	2.574	2.304	4.001	3.079	2.698	0.025332	0.11917	0.2004
48	3.191	2.565	2.295	3.987	3.066	2.685	0.025332	0.11925	0.2006
50	3.183	2.557	2.286	3.975	3.054	2.674	0.025331	0.11932	0.2008
60	3.150	2.525	2.254	3.925	3.008	2.627	0.025329	0.119611	0.201658
70	3.128	2.503	2.231	3.890	2.975	2.595	0.025327	0.119820	0.202288
80	3.111	2.486	2.214	3.864	2.950	2.571	0.025326	0.119978	0.202767
90	3.098	2.473	2.201	3.844	2.932	2.552	0.025325	0.120101	0.203140
100	3.087	2.463	2.191	3.828	2.917	2.537	0.025324	0.120200	0.203442
120	3.072	2.447	2.175	3.805	2.894	2.515	0.025323	0.120349	0.203897
150	3.056	2.432	2.160	3.781	2.872	2.494	0.025322	0.120499	0.204355
180	3.046	2.422	2.149	3.766	2.858	2.479	0.025321	0.120599	0.204666
200	3.041	2.417	2.144	3.758	2.850	2.472	0.025321	0.120649	0.204816
240	3.033	2.409	2.136	3.746	2.839	2.461	0.025320	0.120725	0.205049
300	3.026	2.402	2.129	3.735	2.829	2.451	0.025320	0.120800	0.205282
360	3.021	2.397	2.124	3.727	2.822	2.443	0.025320	0.120851	0.205437
400	3.018	2.394	2.121	3.723	2.818	2.440	0.025319	0.120876	0.205516
480	3.015	2.391	2.117	3.717	2.813	2.435	0.025319	0.120914	0.205633
500	3.014	2.390	2.117	3.716	2.811	2.434	0.025319	0.120922	0.205657
600	3.011	2.387	2.114	3.711	2.807	2.429	0.025319	0.120952	0.205751
720	3.008	2.384	2.111	3.708	2.804	2.422	0.025319	0.120978	0.205830
900	3.0057	2.3818	2.1086	3.7041	2.8000	2.4222	0.025319	0.121003	0.205909
1 000	3.0048	2.3809	2.1077	3.7025	2.7986	2.4208	0.025318	0.121013	0.205940
10^4	2.9966	2.3728	2.0995	3.6902	2.7871	2.4095	0.025318	0.121096	0.206195
10^5	2.9958	2.3720	2.0987	3.6890	2.7860	2.40836	0.025318	0.121104	0.206221
10^6	2.99574	2.37194	2.09861	3.68889	2.7858	2.40824	0.025318	0.121105	0.206224
∞	2.9957322	2.3719323	2.0985978	3.6888794	2.78582183	2.4082292	0.0253178	0.1211046	0.2062240